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Auctions with quantity externalities and endogenous supply[☆]

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ABSTRACT

This paper studies the design of license auctions when the number of licenses allocated in the auction determines structure of the post-auction market. I first show that a sequence of conditional reserve prices that specify minimum acceptable bid at each supply level can be used to determine supply endogenously. Then I construct a static auction called multi-dimensional uniform-price auction and a dynamic auction called Walrasian clock auction that allow the auctioneer to condition reserve price on supply and allow bidders to condition bids on supply. I show that both proposed auctions can implement the efficient market structure that maximizes total surplus in the post-auction market in a dominant strategy equilibrium. I next characterize the optimal auction and show that the two proposed auctions can yield the optimal revenue under a sequence of optimal reserve prices.

1. Introduction

The classical auction theory literature studies the problem of allocating an exogenously given set of items, assuming that bidders' valuations over each item or subset of items are exogenously given, and the final allocation only enters payoffs of auction participants and does not enter payoffs of any outsider who does not participate in the auction. However, these assumptions can fail to hold in many auction markets. This paper studies the design of efficient and optimal auctions when each bidder's value from winning is decreasing in the total quantity of items allocated in the auction, and the total quantity of items allocated in the auction enters payoffs of outsiders who do not participate in the auction. Moreover, the auctioneer not only cares about surplus created within the auction, but also cares about surplus of outsiders who are affected by the auction outcome. A few examples that motivate this research are described below.

Example A (Government-sponsored Auctions for Production Rights)

In many countries, governments use auctions to sell production rights that determines the market structure of the corresponding industry.[1] For example, the US government uses auctions to allocate spectrum licenses to mobile service providers.[2] In this setting, a firm has to obtain a license to enter the industry, and such licenses can only be obtained

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[1] The use of government-sponsored auctions for allocating production rights is discussed in Dana and Spier (1994) and Jehiel and Moldovanu (2004).

[2] The presence of quantity externalities in spectrum auctions is pointed out in Izmalkov et al. (2016). In reality, spectrum auctions usually involve allocation of heterogeneous licenses to multi-unit demand bidders. This paper abstracts away from more complicated spectrum auction models and focuses

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through auctions. The value of a license to each firm depends on the total number of licenses allocated in the auction, as each firm's profit upon entry depends on the number of competitors who also win a license to enter the industry. Selling more licenses allows more firms to enter the market but reduces the payoff from winning a license to each firm. At the same time, consumers can benefit from having more firms competing in the industry, so selling more licenses also increases consumer surplus. The government often cares about efficiency and revenue generated within auction as well as consumer surplus in the post-auction market.

Example B (Car License Plate Auctions in Shanghai)

The Shanghai government uses auctions to sell car license plates to residents who want to purchase new cars for private use in Shanghai, China. A car buyer has to obtain a license plate in order to enter the road, and car license plates can only be obtained through auctions. In this auction, the value of a license plate to each car buyer is decreasing in the total number of license plates allocated in the auction,[3] as selling more licenses allows more cars to enter the road and leads to worse traffic congestion, which reduces each car buyer's value of owning a car. Selling more licenses also increases demand for new cars, leading to higher equilibrium prices in Shanghai's automobile market, further reducing each individual car buyer's surplus, while increasing car sellers' surplus at the same time. The government cares about both car buyers' surplus and car sellers' surplus and therefore faces a trade-off when deciding the number of car license plates to allocate in the auction.

Example C (Sponsored Advertising Auctions in Online Platforms)

Many online platforms such as Amazon.com and eBay.com use auctions to allocate sponsored advertising slots to sellers. A sponsored advertising slot can be viewed as a license for entering the sponsored product list. The value of a slot is decreasing in the total number of slots allocated in the auction, as the click-through rate and conversion rate of each sponsored product are likely to be lower when there are more sponsored products listed on the platform.[4] At the same time, consumers shopping on the platforms may benefit from having more products in the sponsored product list, as having more products on the sponsored list not only provides more choices to the consumers, but also promotes price competition among suppliers. The platform designer cares about both revenue generated from advertisers and surplus of consumers shopping on the platform.

In all the aforementioned examples, auctions are used to sell licenses that are effectively entry permits to some post-auction markets. The number of licenses sold in the auction determines the number of participants on one side of the post-auction market. When more licenses are sold in the auction, more bidders will be competing on the same side of the post-auction market, which decreases the payoff of each individual bidder who enters the market and increases the payoffs of participants on the other side of the post-auction market. Therefore, two types of "quantity externalities" present in the auction. First, selling an additional license imposes negative externalities on all winning bidders, as each bidder's value from winning is decreasing in the total number of licenses allocated in the auction. Second, selling an additional license imposes positive externalities on auction outsiders who participate on the other side of the post-auction market.

Under the presence of these quantity externalities, the auctioneer faces a trade-off when selecting the total number of licenses supplied in the auction. Selling all licenses up to the capacity constraint may not maximize total surplus in the post-auction market. Therefore, it would be natural for the auctioneer to choose the quantity supplied endogenously within auction. How to design an auction to select both the quantity of licenses to sell and the assignment of licenses to bidders is an interesting and challenging problem. On one hand, bidders can fail to consider the negative externalities imposed on other bidders when bidding for licenses. This may result in overbidding and oversupply of licenses, causing too many bidders entering the post-auction market. On the other hand, bidders can fail to take the positive externalities imposed on outsiders into account. This can result in underbidding and undersupply of licenses, causing too few bidders entering the post-auction market. This paper shows that both type of quantity externalities can be "priced" correctly by designing two auctions that can implement the efficient market structure under the presence of quantity externalities.

In this paper, I revisit the model in Gebhardt and Wambach (2008) where a license auction is used for selling entry permits to a post-auction market. In this model, each bidder's value from winning a license depends on its private type and final supply of the auction. Moreover, the final supply of auction also affects payoffs of some auction outsiders who participate on the other side of the post-auction market. I first show that a sequence of conditional reserve prices can be used to determine supply endogenously in the auction. That is, the auctioneer conditions reserve price on final supply by setting a minimum acceptable bid for every additional unit to be sold. For every possible supply level k, a kth unit is sold only if the kth highest bid meets the conditional reserve price r_k. Then I construct a static auction called multi-dimensional uniform-price auction that allows the auctioneer to condition reserve price on supply and allows bidders to condition bids on supply. I show that this auction can implement the efficient allocation in a dominant strategy equilibrium under a sequence of efficient conditional reserve prices. Moreover, I construct a dynamic auction called Walrasian clock auction that can dynamically implement the efficient allocation. With efficient conditional reserve prices, both the multi-

on designing efficient mechanisms under the presence of quantity externalities, with the simplifying assumptions of homogeneous licenses and single-unit demands. How to design a spectrum license auction under quantity externalities with more practical assumptions is left for future research.

[3] The number of licenses should be measured in thousands in this example. Although selling one extra license plate on the margin does not make a difference on traffic congestion or prices in automobile market, selling thousands more licenses can have an impact.

[4] The presence of allocative externalities on sellers in sponsored advertising is pointed out in Hummel and McAfee (2014), Ghosh and Mahdian (2008), and Izmalkov et al. (2016) with different modeling approaches and research focuses. None of these papers considers the potential externalities on consumers.

dimensional uniform-price auction and the Walrasian clock auction are outcome equivalent to the VCG mechanism. I also characterize the revenue-maximizing mechanism and the optimal conditional reserve prices. Both the multi-dimensional uniform-price auction and the Walrasian clock auction can also implement the optimal revenue under optimal conditional reserve prices.

The two proposed auctions have many desirable properties to the auctioneer. First, conditioning reserve price on final supply facilitates price discovery process at every feasible supply level and allows the auctioneer to easily evaluate the marginal benefit and marginal cost of selling every additional license given any bid profile. The auctioneer simply needs to compare the kth highest bid to the conditional reserve price r_k to decide whether a kth license should be allocated. This feature makes the outcome determination straightforward and allows the auction to conclude in a timely manner. Second, both auctions have very little informational requirement on the auctioneer. The auctioneer can set the efficient conditional reserve price schedule without having any information about the distribution of bidders' private types.[5]

In addition to having desirable simplicity and efficiency properties to the auctioneer, the two proposed auctions also have many desirable properties to the bidders. First, bidders are able to protect themselves from instances where the market is over-flooded with supply, making the ex-post payoff from winning significantly lower than the expected payoff at the time of bidding. At the same time, bidders are also able to protect themselves from losing after underbidding and then realizing that they could have received higher payoffs from winning than what they expected. Allowing bidders to condition bids on final supply effectively removes any potential strategic considerations that bidders might otherwise have in order to prevent overpayment when too many licenses are sold or underbidding when too few licenses are sold. Second, both auctions have little informational requirement on bidders, as truthful bidding at every supply level is a dominant strategy equilibrium in both auctions. This feature greatly increases the simplicity of auction design and reduces uncertainty in the auction.

The main contribution of this paper is to construct both static and dynamic auction mechanisms to implement the efficient market structure under the presence of quantity externalities. With correctly designed efficient conditional reserve prices, the two proposed auctions are outcome equivalent to the Jumping English auction proposed in Gebhardt and Wambach (2008) but have simpler design and take shorter time to conclude. Moreover, the two proposed auctions can implement the VCG outcome in a dominant strategy equilibrium, contributing to the literature on practical implementation of the VCG mechanism. Under optimal reserve prices, the two proposed auctions can also yield the optimal revenue, contributing to the literature on optimal auction design. The main implication of this paper is that auctions can be used as regulation devices in markets where firms fail to consider negative externalities on competitors or positive externalities on consumers when making entry decisions, leading to market failure in firm entry and product selection. By imposing a reserve price at every supply level and allowing bidders to condition bids on supply, the auctioneer can price both negative and positive externalities associated with entry correctly under either surplus-maximizing or revenue-maximizing objective.

2. Related Literature

This paper is built upon the strand of literature on designing a private industry using government auctions. Dana and Spier (1994) study mechanisms for selling production rights in which both the market structure and the winners are a function of the bids. They demonstrate that the optimal mechanism is one in which the market structure is endogenous. Jehiel and Moldovanu (2004) examine the existence of mechanisms for implementing efficient market structure in a similar setting and show that efficient auctions exist only if firms' private types are single-dimensional. There are a few other papers that study government-sponsored auctions for selling production rights with different focuses and modeling assumptions (Katz and Shapiro, 1986; Rodriguez, 1997; Jehiel and Moldovanu, 2000; Ozcan, 2004).

Among this strand of literature, this paper is most closely related to Gebhardt and Wambach (2008), who design an efficient license auction for selling entry permits when each bidder's value depends on its private setup cost and the total number of licenses sold in the auction. They propose a dynamic auction called Jumping English auction to maximize total surplus in the post-auction market. In a Jumping English auction with maximum supply level K, the auctioneer specifies a sequence of $2K$ critical price points $(P_1, P^1, P_2, P^2, P_3, P^3, \cdots, P_K, P^K)$.[6] The clock price first increases from P_1 and supply is initially set to 1. If only one bidder is left in the auction before the clock price reaches P^1, then that bidder gets a monopoly license. Otherwise, the clock price jumps down to P_2 and the supply is adjusted to 2, then the price increases again. If there are only two bidders left before the clock price reaches P^2, then those two bidders each get a license. Otherwise, price jumps down to P_3 and supply is adjusted to 3, etc. This process is repeated until the auction reaches a supply level k at which there is no excess demand in the price interval $[P_k, P^k]$. If there is still excess demand at P^K, then the auction ends at P^K with supply level K. A lottery will be used to select K winners from all active bidders at P^K.

While the Jumping English auction can implement the efficient allocation in a dominant strategy equilibrium, it can be difficult to be implemented in practice. First, the use of $2K$ critical price points makes the auction design complicated. Second, because the Jumping English auction consists of a series of price increases (from P_k to P^k) and jumps (from P^k to P_{k+1}) for all $k \in \{1, 2, \cdots, K\}$, it can take a long time for the auction to conclude. This paper extends Gebhardt and Wambach (2008) by proposing two simpler auction mechanisms to implement the efficient market structure in the same

[5] The auctioneer needs to know the distribution of bidders' private types in order to set the optimal conditional reserve prices. This informational requirement is needed for designing any optimal auction with or without quantity externalities.

[6] P_1 is zero in any efficient auction.

setting. More specifically, I show that given any maximum supply level K, only K critical price points (conditional reserve prices) are required to implement the efficient outcome. Then I propose an efficient static auction and an efficient dynamic auction with a sequence of K conditional reserve prices. Compared to the Jumping English auction, these two auctions can implement the VCG outcome in a dominant strategy equilibrium while utilizing fewer critical price points and taking shorter time to conclude. This can be viewed as the main contribution of this paper relative to Gebhardt and Wambach (2008).

This paper is also related to the strand of literature on firm entry and product selection in monopolistic competition markets. Spence (1976) investigates firm entry and product selection in monopolistic competition under entry costs. He shows that monopolistic competition can lead to market failure in product selection due to two forces. First, each firm does not take the impact on consumer surplus into account when making the entry decision. This can cause too few firms entering the market, resulting in too little product diversity. Second, each entering firm does not take adverse effects on competitors' profits into account. This can cause too many firms entering the market, resulting in excessive product diversity. Koenker and Perry (1981) extend the Spence (1976) model and show that excessive diversity is more likely to occur when product differentiation is weak relative to the scale economies of production. This paper contributes to this strand of literature by designing auction mechanisms to implement the socially optimal entry outcome when each individual firm's entry imposes negative externalities on competing firms and positive externalities on consumers. One implication of this paper is that auctions can be used as regulation devices in markets where entry-related externalities lead to market failure. By correctly pricing the externalities associated with entry through a centralized auction mechanism, the regulation authority can implement the socially optimal outcome by selecting both the optimal number of firms to enter the market and the firms that can create greatest surplus upon entry.

Another strand of relevant literature is on auctions with allocative externalities. Jehiel et al. (1996) construct a revenue-maximizing mechanism when a sale creates negative externalities on losing bidders and the magnitude of externalities depends on the identity of the winner. In a subsequent paper, Jehiel et al. (1999) characterize the optimal multi-dimensional mechanism when each buyer's multi-dimensional type specifies its payoff under every possible allocation in the auction. Izmalkov et al. (2016) study sponsored search auctions in which the click-through rate of each advertising slot depends on the total number of advertisements placed on the website. They construct an efficient auction and an optimal auction as direct revelation mechanisms. Other papers in this literature include Varma (2002), Hansen (1988), and Lengwiler (1999). This paper complements this strand of literature by studying a model in which allocative externalities come from how auction shapes the structure of the post-auction market.

3. Model

3.1. Environment

An auctioneer can sell up to K licenses that allow entry to a post-auction market. There are $N \geq K$ bidders hoping to enter the post-auction market.[7] The bidders are indexed by $i \in \{1, 2, \cdots, N\}$. Each bidder is required to have a single license to enter the market and can only obtain a single license through the auction.[8] All bidders who win a license will compete on the same side of the post-auction market. Upon entering the post-auction market, each bidder's payoff $\pi(v_i, n)$ depends on its private type v_i and the total number of entrants n:

$$\pi(v_i, n) = v_i + \delta(n) \quad (1)$$

Assume $\delta(K) = 0$, then v_i can be interpreted as bidder i's payoff from participating in the post-auction market when all K licenses are allocated. $\delta(n)$ can be interpreted as the extra payoff to each entrant when $(K - n)$ fewer bidders are allowed to enter the market. Since each entrant's payoff is lower when there are more competitors, $\delta(1) > \delta(2) > \cdots > \delta(K)$.

Each bidder's value from winning a license equals its payoff from participating in the post-auction market, i.e., each bidder's value from winning in the auction when supply equals n is also given by $\pi(v_i, n)$. Under the context of auctions, $\delta(n)$ represents the premiumness of a license. Each bidder's value of a license is increasing in its private type v_i and decreasing in the total supply n. Assume that $(v_1, v_2, \cdots, v_N)$ are independently and identically distributed according to joint distribution function $F(v)$ with density function $f(v)$ on $[\underline{v}, \bar{v}]$. $\delta(1), \delta(2), \cdots, \delta(K)$ are commonly known to the auctioneer and the bidders.

The number of bidders who enter the post-auction market also affects payoffs of participants on the other side of the market. Let $\pi^o(n)$ denote the payoffs to these auction outsiders when n bidders are allowed to enter. Then

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[7] The number of potential entrants N and the maximum number of licenses to allocate K are exogenous. Note that K represents the maximum supply, not the final supply, in the auction. Given that each bidder only needs a single license to enter the market, it can be assumed that when the number of bidders is lower than the maximum supply ($N < K$), the auctioneer will adjust the maximum supply level to be N. Therefore, we can assume $N \geq K$ here without loss of generality.

[8] In this model, each bidder is restricted to bid for at most one license in the auction. Since each bidder only needs a single license to enter the market, allowing bidders to demand multiple licenses in the auction can only hurt efficiency and increase the complexity of auction design. This is because each bidder's value of a license beyond the first unit is the decrement in payoff associated with having an additional competitor entering the post-auction market. Bidders are unable to take negative externalities on other winning bidders or positive externalities on outsiders into account when bidding for extra licenses, which means that allowing bidders to express their values beyond the first license does not help to improve efficiency. Moreover, since the negative externalities that selling each additional license imposes on each bidder is common information, it is also unnecessary to elicit this information.

$\pi^o(1) < \pi^o(2) < \pi^o(3) < \cdots < \pi^o(K)$, as greater competition on one side increases payoffs of market participants on the other side. Assume that $\pi^o(1), \pi^o(2), \cdots, \pi^o(K)$ are commonly known to the auctioneer and the bidders.

The bidders' value functions and the auction outsiders' payoff functions imply the presence of quantity externalities. For every bidder, obtaining a license imposes negative externalities on all the other winning bidders and imposes positive externalities on some auction outsiders by increasing competition in the post-auction market.

3.2. The auctioneer's surplus-maximization problem

The auctioneer can freely choose to sell any number of licenses up to K licenses. Assume that the auctioneer's objective is to maximize social surplus in the post-auction market. The following analysis shows that in any socially optimal mechanism, both the number of licenses to allocate and the corresponding set of winning bidders should be selected based on the bidders' private types $(v_1, v_2, \cdots, v_N)$ elicited in the mechanism.

Given any private type profile $v \equiv (v_1, v_2, \cdots, v_N)$, let $v_{(n)}$ denote the nth highest value among $(v_1, v_2, \cdots, v_N)$, then $v_{(1)} \geq v_{(2)} \geq v_{(3)} \geq \cdots \geq v_{(N)}$[9]. The social surplus generated by selling n licenses to a set of $S_n \subseteq \{1, 2, \cdots, N\}$ bidders is given by

$$TS(v, n, S_n) = \pi^o(n) + n\delta(n) + \sum_{i \in S_n} v_i \tag{2}$$

The total surplus conditional on selling n licenses is maximized when the n licenses are sold to the n highest-type bidders:

$$TS^*(v, n) = \pi^o(n) + n\delta(n) + \sum_{i=1}^{n} v_{(i)} \tag{3}$$

The socially optimal supply level n^* under type profile v is given by

$$n^* = argmax_n TS^*(v, n) \tag{4}$$

Given any type profile $(v_1, v_2, \cdots, v_N)$, suppose the auctioneer always sells a license to each of the n highest-type bidders conditional on any supply level n, then the social marginal benefit of selling an nth license under any realization of $v_{(n)}$ is given by

$$MB(v_{(n)}, n) = \underbrace{[\pi^o(n) - \pi^o(n-1)]}_{\text{positive externalities}} + \underbrace{v_{(n)} + \delta(n)}_{\text{value to the marginal bidder}} \tag{5}$$

in which $[\pi^o(n) - \pi^o(n-1)]$ is the positive externalities imposed on the outsiders by selling the nth license, and $v_{(n)} + \delta(n)$ is the payoff to the marginal bidder who would not win a license if supply is restricted to $(n-1)$. Assume that $\pi^o(1), \pi^o(2), \cdots, \pi^o(K)$ satisfy the following assumption:

Assumption 1. $\pi^o(n) - \pi^o(n-1) > \pi^o(n+1) - \pi^o(n), \ \forall n \in \{1, 2, \cdots, K-1\}$

Assumption 1 implies that the magnitude of positive externalities from selling an additional license becomes lower when more licenses are sold. Since the marginal bidder's value of winning also becomes lower when more licenses are sold, we must have $MB\big(v_{(n)}, n\big) > MB\big(v_{(n+1)}, n+1\big)$ given any type profile $(v_1, v_2, \cdots, v_N)$, for any n. The social marginal benefit of selling an additional license is diminishing in the number of licenses sold.

On the other hand, the social marginal cost of selling an nth license is given by

$$MC(n) = \underbrace{(n-1)[\delta(n-1) - \delta(n)]}_{\text{negative externalities}} \tag{6}$$

$(n-1)[\delta(n-1) - \delta(n)]$ is the negative externalities imposed on the $(n-1)$ bidders who would still win a license if supply is restricted to $(n-1)$. For each of these bidders, the value from winning becomes lower when the nth license is sold. Assume that $\delta(1), \delta(2), \cdots, \delta(K)$ satisfy the following assumption:

Assumption 2. $(n-1)[\delta(n-1) - \delta(n)] < n[\delta(n) - \delta(n+1)], \ \forall n \in \{1, 2, \cdots, K-1\}$

Assumption 2 implies that the magnitude of negative externalities from selling an additional license becomes greater when more licenses are sold, i.e., $MC(n) < MC(n+1)$ for any n. The social marginal cost of selling an additional license is increasing in the number of licenses sold.

In an efficient license auction that maximizes social surplus in the post-auction market, an nth license should be sold if and only if $MB(v_{(n)}, n) \geq MC(n)$. Given that $MB(v_{(n)}, n)$ is decreasing in n and $MC(n)$ is increasing in n under any type profile $(v_1, v_2, \cdots, v_N)$, the social surplus $TS^*(v, n)$ is maximized at $n = n^*$ such that

$$\begin{aligned} &MB\big(v_{(n)}, n^*\big) \geq MC(n^*) \\ &MB\big(v_{(n+1)}, n^*+1\big) < MC(n^*+1) \end{aligned} \tag{7}$$

[9] Any tie is broken randomly.

This characterization gives the definition of efficient auction.

Definition 1. A license auction is efficient if given any type profile $(v_1, v_2, \cdots, v_N)$, it always allocates a license to each of the n^* highest type bidders, in which

$$n^* = \max\left\{n \middle| MB\left(v_{(n)}, n\right) \geq MC(n), n \in \{1, 2, \cdots, K\}\right\} \tag{8}$$

To make the problem more interesting, assume that the bidders' type space $[\underline{v}, \bar{v}]$ satisfies the following assumption:

Assumption 3. For every $n \in \{2, 3, \cdots, K\}$, there exists a continuum of types $[\underline{v}, v_n^e)$ such that for any $v_i \in [\underline{v}, v_n^e)$,

$$MB(v_i, n) < MC(n) \tag{9}$$

That is, for every unit of license beyond the first unit, there exists a continuum of types $[\underline{v}, v_n^e)$ such that if the nth highest type $v_{(n)}$ falls into this interval, it is not efficient to allocate the nth license. Since $MB(v_{(n)}, n)$ is monotonically increasing in $v_{(n)}$, v_n^e is effectively the minimum acceptable value that the nth highest type can take for n licenses to be allocated in any efficient auction. In other words, v_n^e is the "efficient reserve type" for selling an nth license to be socially beneficial.

Lemma 1. *The efficient reserve types $(v_1^e, v_2^e, \cdots, v_K^e)$ in which v_n^e specifies the minimum value that the nth highest type $v_{(n)}$ can take for an nth license to be sold in any efficient auction are characterized below:*

$$v_n^e = \begin{cases} 0 \quad \text{for} \quad n = 1 \\ \underbrace{(n-1)[\delta(n-1) - \delta(n)]}_{\text{negative externalities}} - \underbrace{[\pi^o(n) - \pi^o(n-1)]}_{\text{positive externalities}} - \delta(n) \quad \text{for all} \quad n \in \{2, 3, \cdots, K\} \end{cases} \tag{10}$$

For every unit n beyond the first unit, the efficient reserve type v_n^e that justifies selling the nth license to the nth highest type bidder is the threshold type that satisfies $MB(v_n^e, n) = MC(n)$. Since there is no social marginal cost associated with selling the first license, the efficient reserve type for the first license is always zero. For every $n \in \{2, 3, \cdots, K\}$, the allocation of an nth license imposes negative externalities on all the $(n-1)$ highest-type bidders who would still win a license when the nth license is not allocated. Whether it worths to allocate the nth license depends on whether the social marginal benefit of selling an nth license exceeds the associated social marginal cost. Since allocating an nth license to a marginal bidder with a type below v_n^e makes $MB(v_{(n)}, n) < MC(n)$, it is not efficient to allocate an nth license if the nth highest type $v_{(n)}$ fails to meet the efficient reserve type v_n^e. Therefore, under any exogenously given maximum supply level K, the final supply of licenses should always be chosen endogenously in an efficient auction, and a sequence of positive reserve prices can be used to withheld items efficiently.

Note that the positive term in v_n^e is increasing in n, while all negative terms in v_n^e are decreasing in n, which implies that the efficient reserve types $(v_1^e, v_2^e, \cdots, v_K^e)$ must satisfy $0 = v_1^e < v_2^e < \cdots < v_K^e$. That is, the minimum acceptable type for the highest type bidder to enter the market is lower than the minimum acceptable type for the second highest type bidder to enter the market, which is in turn lower than the minimum acceptable type for the third highest type bidder to enter the market, etc. As more licenses are allocated, the minimum acceptable type of the marginal bidder becomes greater.

Moreover, note that all terms in v_n^e are common knowledge in the game. This gives an informational advantage to the auctioneer, as it effectively allows the auctioneer to set reserve prices to withhold licenses efficiently without knowing bidders' private types.

3.3. The VCG mechanism

I next describe the VCG mechanism under this context and prove that the VCG mechanism implements the efficient allocation in a dominant strategy equilibrium.

Definition 2. In a VCG mechanism, each bidder is asked to report its private type v_i. Given any profile of reported types $\hat{v} \equiv (\hat{v}_1, \hat{v}_2, \cdots, \hat{v}_N)$, the auctioneer ranks the reports in descending order, $\hat{v}_{(1)} \geq \hat{v}_{(2)} \geq \cdots \geq \hat{v}_{(N)}$, and then allocates licenses according to the following algorithm:

(R1) Allocate one license to the bidder with the highest reported type $\hat{v}_{(1)}$ and continue to (R2).

(R2) Allocate one license to the bidder with the second highest reported type $\hat{v}_{(2)}$ and continue to (R3) if $\hat{v}_{(2)} \geq v_2^e$. Restrict final supply to 1 license otherwise.

(R3) Allocate one license to the bidder with the third highest reported type $\hat{v}_{(3)}$ and continue to (R4) if $\hat{v}_{(3)} \geq v_3^e$. Restrict final supply to 2 licenses otherwise. $\cdots$

(RK) Allocate one license to the bidder with the Kth highest reported type $\hat{v}_{(K)}$ if $\hat{v}_{(K)} \geq v_K^e$. Restrict final supply to $(K-1)$ licenses otherwise.

Given any final supply level n determined by the above algorithm, each winning bidder's VCG payment is given by

$$p^n(\hat{v}) = \max\left\{\hat{v}_{(n+1)},\ v_n^e\right\} + \delta(n) \tag{11}$$

Each losing bidder's VCG payment is zero.

In a VCG mechanism, each agent is required to pay the externalities it imposes on other agents in the society. In standard auctions without quantity externalities,[10] each winning bidder's VCG payment is given by the value of the marginal losing bidder. Under the presence of quantity externalities, the externalities that each winning bidder imposes on other agents in the society becomes ambiguous. Since the auctioneer wants to maximize social surplus in the post-auction market, both externalities imposed on other bidders and externalities imposed on participants on the other side of the post-auction market should be considered when calculating each winning bidder's VCG payment.

Given any supply level n, there are two types of externalities that each winning bidder can impose on the society. First, each winning bidder can impose externalities on losing bidders by depriving a license from the marginal losing bidder. Second, each winning bidder can impose externalities on other winning bidders and participants on the other side of the post-auction market by increasing the final supply level from $(n-1)$ to n. Note that the first type of externalities presents in any standard auction setting, while the second type of externalities only presents under quantity externalities. Since the magnitudes of both externalities do not depend on the type of the winning bidder, every winning bidder's VCG payment should be the same under any supply level n.

Given the two different types of externalities that each winning bidder can impose on the society, which externalities should be charged as the VCG payment depends on what would happen if any of the n winning bidders was absent in the auction. When $\hat{v}_{(n+1)} \geq v_n^e$, the marginal losing bidder would win a license absent the presence of any winning bidder, implying that the first type of externalities is in effect, and each winning bidder's VCG payment should be given by $\hat{v}_{(n+1)} + \delta(n)$, the negative externalities imposed on the marginal bidder. When $\hat{v}_{(n+1)} < v_n^e$, the final supply would be $(n-1)$ instead of n absent the presence of any winning bidder, implying that the second type of externalities is in effect, and each winning bidder's VCG payment should be given by $(n-1)[\delta(n-1)-\delta(n)]-[\pi^o(n)-\pi^o(n-1)]$, the negative quantity externalities imposed on other winning bidders subtracted by the positive quantity externalities imposed on auction outsiders. Note that the VCG payment given in Eq. (11) can be rewritten as

$$p^n(\hat{v}) = \max\left\{ \underbrace{\hat{v}_{(n+1)} + \delta(n)}_{\text{externalities on marginal bidder}}, \quad \underbrace{(n-1)[\delta(n-1)-\delta(n)]}_{\text{negative quantity externalities}} - \underbrace{[\pi^o(n)-\pi^o(n-1)]}_{\text{positive quantity externalities}} \right\} \tag{12}$$

The VCG payment rule effectively "prices" the quantity externalities present in the auction and ensures that each winner's payment is no lower than the quantity externalities incurred from winning an license.

It can be shown that truth-telling by reporting true types is a dominant strategy for every bidder in the VCG mechanism under quantity externalities.

Proposition 1. *Truth-telling is a dominant strategy equilibrium in the VCG mechanism.*[11]

Proof. See Appendix. □

4. Two practical efficient auctions

Despite of its efficiency, the VCG mechanism is rarely implemented in practice, as it can be impractical to ask bidders report their types directly in real-world auctions.[12] In this section, I propose two practical non-direct mechanisms that are outcome equivalent to the VCG mechanism under correctly designed reserve prices.

Lemma 1 implies that final supply should be determined endogenously based on the bid profile in any efficient auction under the presence of quantity externalities.[13] A natural instrument to implement this goal is to use a sequence of conditional reserve prices to determine supply endogenously. That is, for every feasible supply level $n \in \{1, 2, \cdots, K\}$, the auctioneer sets a conditional reserve price $r_n \geq 0$ such that the nth license is allocated if and only if the nth highest bid is no lower than r_n. Since the social marginal cost of selling the first license is zero and the social marginal benefit of selling the first license is strictly positive, the conditional reserve price for selling the first license is always zero in any efficient auction.

For every additional unit $n \in \{2, 3, \cdots, K\}$ beyond the first license, the conditional reserve price r_n must be chosen such that any bidder with type below the efficient reserve type v_n^e bids below r_n, and any bidder with type above the efficient reserve type v_n^e bids above r_n in equilibrium. This property ensures that an nth license is always allocated when the nth highest type $v_{(n)}$ is above the efficient reserve type v_n^e, implying that social marginal benefit of selling an nth license is greater than the social marginal cost, and ensures that an nth license is always withheld by the auctioneer when the nth highest type $v_{(n)}$ falls below the efficient reserve type v_n^e, implying that social marginal benefit of selling an nth license is lower than the social marginal cost. Since the social marginal benefit of selling an additional license is decreasing in n

[10] Assuming homogeneous items and single-unit demand.

[11] This proposition is a corollary to Gebhardt and Wambach (2008)'s discussion on the VCG mechanism with omitted proof.

[12] In this setting, the auctioneer may elicit bidders' type profiles $(v_1, v_2, \cdots, v_N)$ by committing to selling K licenses and then running a standard uniform-price auction. However, it is unethical for the auctioneer to renege on the supply level it commits to when the elicited type profile implies that efficient supply level should be lower than K.

[13] This is also pointed out in Dana and Spier (1994).

and the social marginal cost of selling an additional license is increasing in n, we must have $0 = r_1 < r_2 < \cdots < r_K$, i.e., the minimum acceptable bid for each additional license becomes higher when more licenses are allocated.

Lemma 2. *In any efficient auction that adopts a sequence of conditional reserve prices* $(r_1, r_2, \cdots, r_K)$ *where* r_n *specifies the minimum acceptable bid at supply level n, the reserve prices must satisfy*

$$0 = r_1 < r_2 < r_3 < \cdots < r_K \tag{13}$$

For any $n \in \{1, 2, \cdots, K\}$, r_n *must satisfy*

$$\begin{aligned} \beta^n(v_i) < r_n, \quad \forall v_i < v_n^e \\ \beta^n(v_i) \geq r_n, \quad \forall v_i \geq v_n^e \end{aligned} \tag{14}$$

in any auction with monotonic equilibrium bid strategy $\beta^n(.)$ *at supply level n.*

In this section, I construct two auctions to maximize total surplus in the post-auction market by adopting such sequence of reserve prices. Both proposed auctions can implement the VCG outcome in a dominant strategy equilibrium with correctly specified efficient conditional reserve prices.

4.1. Multi-dimensional uniform-price auction

This subsection constructs a static sealed-bid auction called multi-dimensional uniform-price auction that allows the auctioneer to condition reserve price on supply and allows the bidders to condition bids on supply.

Definition 3. In a multi-dimensional uniform-price auction, the auctioneer announces a sequence of conditional reserve prices $(r_1, r_2, \cdots, r_K)$ where $r_1 < r_2 < r_3 \cdots < r_K$. Each bidder $i \in N$ submits a vector of bids $b_i = (b_i^1, b_i^2, \cdots, b_i^K) \in \mathbb{R}^K$, in which b_i^n is bidder i's bid conditional on final supply being equal to n. The auctioneer processes bids in a sequence of rounds. For every possible supply level $n \in \{1, 2, \cdots, K\}$, let S_n denote the set of bidders who win a license in or before round (Rn).

(R1) Rank all bids in $\{b_i^1\}_{i \in N}$. Allocate one license to the highest bidder and continue to (R2) if $\max\{b_i^1\}_{i \in N} > r_1$, restrict supply to be zero and end the auction otherwise.

(R2) Rank all bids in $\{b_i^2\}_{i \in N \setminus S_1}$. Allocate one license to the highest bidder among the remaining bidders who have not won a license and continue to (R3) if $\max\{b_i^2\}_{i \in N \setminus S_1} > r_2$, restrict supply to 1 and end the auction otherwise. $\cdots$

Rank all bids among remaining bidders conditional on the next higher supply level until reaching the highest integer $n^* \leq K$ s.t. $\max\{b_i^n\}_{i \in N \setminus S_{n-1}} > r_n$. Then n^* is the final supply level chosen in the auction.

For any final supply level n^*, each winner's payment is given by

$$p^n(b) = \max\left\{r_{n^*}, \hat{b}^{n^*}\right\} \tag{15}$$

in which $\hat{b}^{n^*}$ is the highest losing bid in $\{b_i^{n^*}\}_{i \in N \setminus S_{n^*-1}}$.

The following examples illustrate how the multi-dimensional uniform-price auction works.

Example 1. Consider an auction with $N = 3$ bidders, A, B, and C. The auctioneer can allocate up to $K = 2$ licenses. In a multi-dimensional uniform-price auction with conditional reserve prices $(r_1, r_2) = (0, 5)$, suppose the bid profile is given as follows:

	Bidder A	Bidder B	Bidder C
b_i^1	10	8	3
b_i^2	6	5	2

The auctioneer processes bids in the following procedure:

(R1) Rank all bids conditional on supply being equal to 1. Since $b_A^1 > b_B^1 > b_C^1$ and $b_A^1 \geq r_1$, allocate one license to bidder A and continue to the next round.

(R2) Rank all bids conditional on supply being equal to 2 among the remaining bidders who have not won a license. Since $b_B^2 > b_C^2$ and $b_B^2 \geq r_2$, allocate one license to bidder B. The final supply is therefore set at 2.

Both Bidder A and Bidder B's payment is given by $p^2(b) = \max\{r_2, b_C^2\} = r_2 = 5$.

Example 2. In the previous example, suppose the bid profile is given as follows:

	Bidder A	Bidder B	Bidder C
b_i^1	10	8	3
b_i^2	6	4	2

The auctioneer processes bids in the following procedure:

(R1) Rank all bids conditional on supply being equal to 1. Since $b_A^1 > b_B^1 > b_C^1$ and $b_A^1 \geq r_1$, allocate one license to bidder A and continue to the next round.

(R2) Rank all bids conditional on supply being equal to 2 among remaining bidders who have not won a license. Since the highest bid $b_B^2 < r_2$, restrict supply to be 1 and end the auction.

Bidder A's payment is given by $p^1(b) = \max\{r_1, b_B^1\} = b_B^1 = 8$.

4.2. Equilibrium of multi-dimensional uniform-price auction

This subsection characterizes the equilibrium of multi-dimensional uniform-price auctions.

Proposition 2. *Under any sequence of conditional reserve prices $(r_1, r_2, \cdots, r_K)$ that satisfy $r_1 < r_2 < r_3 < \cdots < r_K$, a symmetric monotonic equilibrium bidding strategy $\beta(.) \equiv (\beta^1(.), \beta^2(.), \cdots, \beta^K(.))$ exists in the multi-dimensional uniform-price auction. Each bidder i's dominant strategy $b_i^* = \beta(v_i)$ is characterized below.*

For all $n \in \{1, 2, \cdots, K\}$,

$$\beta^n(v_i) = \begin{cases} v_i + \delta(n), & \text{for} \quad v_i \in \left[r_n - \delta(n), \bar{v}\right] \\ 0, & \text{for} \quad v_i \in \left[\underline{v}, r_n - \delta(n)\right) \end{cases} \tag{16}$$

The N-tuple strategy $(b_1^*, b_2^*, \cdots, b_N^*)$ where $b_i^* = \beta(v_i)$ is a dominant strategy equilibrium in the multi-dimensional uniform-price auction.

Proof. See Appendix. □

The dominant strategy equilibrium in the multi-dimensional uniform-price auction has the following monotonicity properties. First, every bidder's equilibrium bid conditional on a lower supply level is always greater than its equilibrium bid conditional on a higher supply level, i.e.,

$$\beta^n(v_i) \geq \beta^{n+1}(v_i), \quad \forall n \in \{1, 2, \cdots, K-1\}, \quad \forall v_i \in [\underline{v}, \bar{v}] \tag{17}$$

Second, the equilibrium bidding strategy $\beta^n(v_i)$ conditional on any supply level n is monotonically increasing in each bidder's private type v_i, i.e.,

$$\beta^n(v_i) \geq \beta^n(v_j) \quad \text{if and only if} \quad v_i \geq v_j, \quad \forall n \in \{1, 2, \cdots, K\} \tag{18}$$

The second monotonicity property also implies that if $\beta^k(v_i) \geq \beta^k(v_j)$ at some k, then $\beta^n(v_i) \geq \beta^n(v_j)$ for any $n \in \{1, 2, \cdots, K\}$. That is, the ranking of bidders at their equilibrium bids stays the same across different supply levels.

Comparing Proposition 2 and Lemma 2 shows that the multi-dimensional uniform-price auction is efficient if and only if the dominant strategy equilibrium $\beta(.)$ characterized in Proposition 2 satisfies the following condition:

For all $n \in \{1, 2, \cdots, K\}$,

$$\beta^n(v_i) = \begin{cases} v_i + \delta(n), & \text{for} \quad v_i \in \left[v_n^e, \bar{v}\right] \\ 0, & \text{for} \quad v_i \in \left[\underline{v}, v_n^e\right) \end{cases} \tag{19}$$

in which v_n^e is the efficient reserve type characterized in Lemma 1. Eq. (19) requires that for every possible supply level n, it is a dominant strategy for each bidder to bid its true value when its type is above v_n^e and bid zero otherwise. Comparing Eqs. (16) and (19) shows that the efficient reserve prices should be set at $r_1^e = 0$ and $r_n^e = v_n^e + \delta(n)$ for all $n \in \{2, 3, \cdots, K\}$, which gives the following corollary.

Corollary 1. *The multi-dimensional uniform-price auction implements the efficient allocation in a dominant strategy equilibrium with a sequence of efficient conditional reserve prices $(r_1^e, r_2^e, \cdots, r_K^e)$, where $r_1^e = 0$ and $r_n^e = v_n^e + \delta(n)$ for all $n \in \{2, 3, \cdots, K\}$. More specifically,*

$$r_n^e = \begin{cases} 0, & \text{for} \quad n = 1 \\ \underbrace{(n-1)[\delta(n-1) - \delta(n)]}_{\text{negative externalities}} - \underbrace{[\pi^o(n) - \pi^o(n-1)]}_{\text{positive externalities}}, & \forall n \in \{2, 3, \cdots, K\} \end{cases} \tag{20}$$

Under efficient conditional reserve prices, the multi-dimensional uniform-price auction is outcome equivalent to the VCG mechanism and the Jumping English auction in Gebhardt and Wambach (2008).

Note that the efficient reserve price r_n^e is given by the net quantity externalities associated with selling an nth license. The efficient reserve price schedule can therefore be viewed as an instrument to correctly "price" quantity externalities in the auction. Since both negative externalities and positive externalities are common knowledge, the auctioneer is able to set the efficient reserve price schedule without acquiring any additional information. Given that the negative externalities is increasing in n and the positive externalities is decreasing in n, it is straightforward to see $0 = r_1^e < r_2^e < \cdots r_K^e$.

The efficiency of the multi-dimensional uniform-price auction is driven by the multi-dimensional bidding language and conditional reserve prices. The former allows bidders to condition their bids on supply, and the latter allows the auctioneer to condition reserve prices on supply. By allowing the bidders to condition their bids on supply, the uncertainty in bidders' values at the time of bidding is eliminated, and any strategic considerations under this uncertainty is therefore removed. Each bidder can easily incorporate its values conditional on winning at different supply levels into its bids. For every feasible supply level n, it is a dominant strategy for each bidder to bid its true value conditional on supply level n when its conditional value exceeds the conditional reserve price r_n. This feature allows the auctioneer to elicit bidders' values conditional on every supply level and understand the marginal social benefit of selling every additional license. At the same time, allowing the auctioneer to condition reserve prices on final supply enables the auctioneer to maximize social surplus by comparing the marginal social benefit from selling an additional license to the corresponding marginal social cost at every feasible supply level.

Furthermore, allowing the auctioneer to condition reserve prices on supply and allowing bidders to condition bids on supply are not only desirable but also essential in driving efficiency. Consider an alternative uniform-price auction with a single reserve price. It would be impossible for the auctioneer to choose the socially optimal supply level, as both marginal social benefit and marginal social cost of selling an additional license differ across supply levels, and a single-dimensional reserve price is not sufficient for evaluating trade-offs at every supply level. On the other hand, consider an alternative uniform-price auction where a sequence of conditional reserve prices is used but each bidder can only submit a single-dimensional bid. This alternative auction is inefficient given any conditional reserve prices. This comes from the fact that when final supply is uncertain and only single-dimensional bids are allowed, each bidder bids according to its expected value conditional on winning in equilibrium. A low-type bidder knows that it only wins when the competition is weak, leading to few licenses being allocated. As a result, the expected value conditional on winning can be high for a low-type bidder. A high-type bidder knows that it may still win when the competition is strong within auction, leading to many licenses being allocated. As a result, the expected value conditional on winning can be low for a high-type bidder. Given this differentiated beliefs in expected supply conditional on winning, lower type bidders bid more aggressively than higher type bidders in equilibrium when bidders are not allowed to condition bids on supply, which leads to inefficiency. Therefore, allowing the auctioneer to condition reserve prices on supply and allowing bidders to condition bids on supply are essential features that drive efficiency in the multi-dimensional uniform-price auction.

Compared to the Jumping English auction in Gebhardt and Wambach (2008) that is also outcome equivalent to the VCG mechanism, the multi-dimensional uniform-price auction has the following desirable properties. First, its static nature makes it practically easier to be implemented. Dynamic auctions requires gathering all bidders to participate in a centralized clock auction at the same time, which is usually not feasible in practice. Second, its sealed-bid nature better preserves bidders' privacy. Third, its static nature makes it take shorter time to conclude than dynamic auctions. Therefore, the multi-dimensional uniform-price auction can be viewed as a more practical mechanism that implements the VCG outcome.

4.3. Walrasian clock auction

This subsection constructs a dynamic auction called Walrasian clock auction, in which the clock price can either go up or go down depending on whether there is excess demand or excess supply. The Walrasian clock auction is defined as follows.

Definition 4. In a Walrasian clock auction, there is a clock showing the current price. At any time of the auction, each bidder states whether it is "in" or "out" of the auction given the current clock price. Denote the number of active bidders at clock price p as $k(p)$, then $k(p)$ represents the aggregate demand at price p. At the beginning of the auction, the auctioneer announces a sequence of reserve prices $(r_1, r_2, \cdots, r_K)$ where $r_1 < r_2 < \cdots < r_K$. The auction proceeds in a sequence of rounds described below:

(RK) The auctioneer sets supply to be K and sets clock price to be r_K. Each bidder states whether it is "in" or "out" of the auction at r_K. The auctioneer then compares aggregate demand $k(r_K)$ to aggregate supply K.

- If $k(r_K) > K$, there is excess demand at price r_K. The auctioneer sets final supply to be K and then runs a standard ascending clock auction with reserve price r_K.
- If $k(r_K) = K$, then the auction ends immediately. All the K active bidders receive a license at price r_K.
- If $k(r_K) < K$, there is excess supply at price r_K. The auctioneer reduces supply to $K-1$ and reduces clock price to r_{K-1}. The auction continues to round (RK-1).

(RK-1) The auctioneer sets supply to be $K-1$ and sets clock price to be r_{K-1}. Each bidder states whether it is "in" or "out" of the auction at r_{K-1}. Those who are active at price r_K in the previous round (RK) are required to stay in the auction when price jumps down to r_{K-1}, so $k(r_{K-1}) \geq k(r_K)$. The auctioneer then compares aggregate demand $k(r_{K-1})$ to aggregate supply $K-1$.

- If $k(r_{K-1}) > K-1$, there is excess demand at price r_{K-1}. The auctioneer sets final supply to be $K-1$ and then runs an ascending clock auction with reserve price r_{K-1}.
- If $k(r_{K-1}) = K-1$, the auction ends immediately. All the $K-1$ active bidders receive a license at price r_{K-1}.
- If $k(r_{K-1}) < K-1$, there is excess supply at price r_{K-1}. The auctioneer reduces supply to $K-2$ and reduce clock price to r_{K-2}. The auction continues to round (RK-2). ...

(Rn) The auctioneer sets total supply to be n and set clock price to be r_n. Each bidder states whether it is "in" or "out" of the auction. Those who are active at price r_{n+1} in the previous round (Rn+1) are required to remain in the auction when price jumps down to r_n. The auctioneer then compares aggregate demand $k(r_n)$ to aggregate supply n.

- If $k(r_n) > n$, there is excess demand at price r_n. The auctioneer sets final supply to be n and runs an ascending clock auction with reserve price r_n.
- If $k(r_n) = n$, the auction ends immediately. All the n active bidders receive a license at price r_n.
- If $k(r_n) < n$, there is excess supply at r_n. The auctioneer reduces supply to $n-1$ and reduces clock price to r_{n-1}. The auction continues to round (Rn-1).

In the Walrasian clock auction, the supply level is adjusted down until reaching the highest integer $n \le K$ such that there is no excess supply at r_n, i.e., $k(r_n) \ge n$. Then the auction turns into a standard ascending clock auction with n licenses and reserve price r_n. If there is still excess supply in the final round (R1), i.e., $k(r_1) = 0$, then the auction ends at the beginning of the final round with no license being sold.

The following examples illustrate how the Walrasian clock auction works.

Example 3. Consider an auction with $N = 3$ bidders, A, B, and C. The auctioneer can allocate up to $K = 2$ licenses. In a Walrasian Clock auction with conditional reserve prices $(r_1, r_2) = (0, 5)$, the auction starts from round (R2), in which the auctioneer sets supply at 2 and sets clock price at 5 at the beginning of the round. Suppose the bid profile is given as follows:

Supply	Price	Bidder A	Bidder B	Bidder C	Excess Supply
2	5	in	in	out	0

Since there are exactly 2 active bidders at $r_2 = 5$ when supply is set at 2, there is no excess supply or excess demand in the auction. The auction ends immediately and the final supply is set at 2. Both Bidder A and Bidder B receive a license at the final clock price of 5.

Example 4. In the previous example, suppose the bid profile in the first round (R2) is given as follows:

Supply	Price	Bidder A	Bidder B	Bidder C	Excess Supply
2	5	in	out	out	1

Since there is only 1 active bidder at $r_2 = 5$, there is excess supply in round (R2). Round (R2) ends immediately and the auction proceeds to the next round (R1). The auctioneer adjusts supply to be 1 and adjusts clock price to be 0. Suppose the bid profile in (R1) is given as follows:

Supply	Price	Bidder A	Bidder B	Bidder C	Excess Demand
1	0	in	in	in	2
1	3	in	in	out	1
1	8	in	out	out	0

Since all 3 bidders are active at $r_1 = 0$ when supply is set at 1, there is excess demand in the auction at the beginning of (R1). The auctioneer sets final supply to be 1 and starts an ascending clock auction from reserve price r_1. When the clock price reaches $p = 3$, Bidder C drops out, which reduces excess demand from 2 to 1. The clock price continues to increase since there is still excess demand. When the clock prices reaches $p = 8$, Bidder B drops out, which further reduces excess demand from 1 to 0 and ends the auction. Bidder A receives a license at the final clock price of 8.

4.4. Equilibrium of Walrasian clock auction

The next proposition characterizes a dominant strategy equilibrium of the Walrasian clock auction:

Proposition 3. *For all feasible supply levels $n \in \{K, K-1, \cdots, 2, 1\}$, a dominant strategy equilibrium in round (Rn) of the Walrasian clock auction is characterized as follows.*

1. *At the beginning of every round (Rn), it is a dominant strategy equilibrium for each bidder i to state "in" if $v_i + \delta(n) \ge r_n$ and state "out" if $v_i + \delta(n) < r_n$. Only bidders with types $v_i \ge r_n - \delta(n)$ are active in round (Rn).*
2. *If $k(r_n) \ge n$ and the auction transforms into an ascending clock auction with n licenses, it is a dominant strategy equilibrium for each bidder who states "in" at price r_n to drop out at its true value of winning one out of n licenses, $v_i + \delta(n)$.*
3. *If $k(r_n) < n$ and the auction proceeds to round (Rn-1), the equilibrium in (Rn) with all n replaced by $(n-1)$ is a dominant strategy equilibrium in round (Rn-1).*

Proof. See Appendix. □

Given the equilibrium characterized above, it is straightforward to see that with reserve prices $r_1^e = 0$ and $r_n^e = v_n^e + \delta(n)$ for $n \in \{2, 3, \cdots, K\}$, the Walrasian clock auction implements the VCG outcome in a dominant strategy equilibrium.

Corollary 2. *The Walrasian clock auction implements the efficient allocation in a dominant strategy equilibrium with a sequence of efficient conditional reserve prices* $(r_1^e, r_2^e, \cdots, r_K^e)$*, where* $r_1^e = 0$ *and* $r_n^e = v_n^e + \delta(n)$ *for all* $n \in \{2, 3, \cdots, K\}$*. More specifically,*

$$r_n^e = \begin{cases} 0, \quad \text{for} \quad n = 1 \\ \underbrace{(n-1)[\delta(n-1) - \delta(n)]}_{\text{negative externalities}} - \underbrace{[\pi^o(n) - \pi^o(n-1)]}_{\text{positive externalities}}, \quad \forall n \in \{2, 3, \cdots, K\} \end{cases} \tag{21}$$

Under efficient conditional reserve prices, the Walrasian clock auction is outcome equivalent to the VCG mechanism, the Jumping English auction in Gebhardt and Wambach (2008), and the multi-dimensional uniform-price auction.

Compared to the Jumping English auction proposed in Gebhardt and Wambach (2008), the Walrasian clock auction have a few advantages. First, it only uses K critical price points to allocate up to K licenses, which simplifies the auction design and makes it easier for practical implementation. Moreover, since each round ends immediately when there is excess supply, and a full ascending clock auction is only conducted in the final round, the Walrasian clock auction takes shorter time to conclude compared to the Jumping English auction, which consists of a sequence of ascending clock auctions before reaching the efficient supply level.

Compared to the multi-dimensional uniform-price auction, the Walrasian clock auction implements the VCG outcome using a dynamic format under the same conditional reserve prices. More interestingly, the Walrasian clock auction implicitly allows bidders to condition their bids on final supply by adjusting supply level at the beginning of every round in the auction. Bidders become certain about final supply once the auction proceeds into the ascending clock stage. The Walrasian clock auction can be viewed as the dynamic format of the multi-dimensional uniform-price auction. The outcome equivalence between the two proposed auctions echos the classical equivalence between second-price auctions and English auctions. Note that the multi-dimensional uniform-price auctions, the Walrasian clock auctions and the Jumping English auctions all allow the auctioneer to condition reserve prices on supply and allow bidders to condition bids on supply. This is the main source that drives efficiency in auctions under quantity externalities.

5. An optimal direct mechanism

In this section, I follow Myerson (1981)'s optimal auction design approach and characterize the optimal auction under quantity externalities as a direct revelation mechanism. I show that the multi-dimensional uniform-price auction and the Walrasian clock auction can also implement the optimal revenue with a sequence of conditional optimal reserve prices.

5.1. Mechanism design and solution concepts

In a direct mechanism, bidders report their private types v_i directly. An auction mechanism (μ, t) consists of an allocation rule $\mu_i(v)$ and a payment rule $t_i(v)$ for every bidder i. $\mu_i = \left(\mu_i^{(1)}(v), \mu_i^{(2)}(v), \cdots, \mu_i^{(K)}(v)\right)$, in which $\mu_i^{(n)}(v)$ is the joint probability that bidder i wins a license when supply equals to n given reported types $v \in [\underline{v}, \bar{v}]^N$. $t_i(v)$ is the expected payment of bidder i given reported types $v \in [\underline{v}, \bar{v}]^N$. The feasibility constraint in a direct mechanism is defined as follows:

Definition 5. An allocation rule μ is feasible if for every supply level $n \in \{1, 2, \cdots, K\}$, given any reported type profile v,

$$(1)\ 0 \le \mu_i^{(n)}(v) \le 1,\ \forall i, \quad \text{and} \quad \sum_i \mu_i^{(n)}(v) \le n \tag{22}$$

(2) If $\mu_i^{(n)}(v) > 0$ for some i, then $\mu_j^{(k)}(v) = 0$, for all $k \neq n$, for all j.

That is, at any supply level n, each bidder's probability of winning must fall in [0,1] and at most n unit can be allocated. The total supply must be unique in the auction.

For each bidder i with type v_i, the interim utility $U_i(v_i)$ is given by

$$U_i(v_i) = \int_{v_{-i}} \underbrace{\left[\sum_{n=1}^{K} \mu_i^{(n)}(v_i, v_{-i})\left[v_i + \delta(n)\right] - t_i(v_i, v_{-i})\right]}_{u_i(v_i, v_{-i})} f_{-i}(v_{-i}) dv_{-i} \tag{23}$$

in which $u_i(v_i, v_{-i}) = \sum_{n=1}^{K} \mu_i^{(n)}(v_i, v_{-i})\left[v_i + \delta(n)\right] - t_i(v_i, v_{-i})$ is the ex-post utility of bidder i given reports (v_i, v_{-i}). A direct mechanism (μ, t) satisfies incentive compatibility and individual rationality conditions if the following definition holds:

Definition 6. A direct auction mechanism (μ, t) is Bayesian incentive compatible (IC) and individually rational (IR) if for every bidder i, given any true type v_i and any reported type v_i',

$$\begin{aligned} U_i(v_i) &\geq \int_{v_{-i}} \left[\sum_{n=1}^{K} \mu_i^{(n)}(v_i', v_{-i})\left[v_i + \delta(n)\right] - t_i(v_i', v_{-i}) \right] f_{-i}(v_{-i}) dv_{-i} \\ U_i(v_i) &\geq 0 \end{aligned} \tag{24}$$

In the following analysis, I characterize the optimal auction mechanism under quantity externalities among all Bayesian IC and IR mechanisms subject to the feasibility constraint.

5.2. Characterization of the optimal auction mechanism

For any possible supply level $n \in \{1, 2, \cdots, K\}$, define bidder i's marginal revenue conditional on total supply being equal to n as

$$MR(v_i, n) = v_i + \delta(n) - \frac{1 - F_i(v_i)}{f_i(v_i)} \tag{25}$$

$MR(v_i, n)$ represents the marginal revenue gain to the auctioneer when it allocates a license to bidder i conditional on supply level being equal to n.

Assume the marginal revenue function $MR(v_i, n)$ is strictly increasing in v_i for all n. The next lemma gives a characterization of any Bayesian IC and IR mechanism.

Lemma 3. *A mechanism (μ, t) is Bayesian IC and IR if and only if for every bidder i, the following conditions hold:*

(1)

For any $v_i, v_i' \in [\underline{v}, \bar{v}]$, if $v_i \geq v_i'$, then

$$\int_{v_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(v_i', v_{-i}) f_{-i}(v_{-i}) dv_{-i} \leq \int_{v_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(v_i, v_{-i}) f_{-i}(v_{-i}) dv_{-i} \tag{26}$$

(2)

$$U_i(v_i) = U_i(\underline{v}) + \int_{\underline{v}}^{v_i} \int_{v_{-i}} \sum_{n=1}^{K} \mu_i^{(n)}(s, v_{-i}) f_{-i}(v_{-i}) dv_{-i} ds \tag{27}$$

(3)

$$U_i(\underline{v}) \geq 0 \tag{28}$$

Proof. See Appendix. □

Lemma 3 shows that in any Bayesian IC and IR mechanism, each bidder's expected probability of winning must be increasing in its true type v_i[14]; each bidder's interim utility by reporting true type v_i is given by the sum of the interim utility of the lowest type bidder and the increase in expected probability of winning from having a type above the lowest type $\underline{v}$; and the interim utility of the lowest type bidder must be weakly greater than zero. The intuition of Lemma 3 is the follows: in order to incentivize each bidder to report its true type (IC), the expected probability of winning should always be increasing in each bidder's type, and each bidder with a type above the lowest type should be rewarded by the increased expected probability of winning in expectation. In order to incentivize all bidders to participate in the mechanism (IR), the expected payoff to the lowest type bidder must be weakly positive.

The next lemma characterizes the seller's ex-ante expected revenue in any Bayesian IC and IR mechanism:

Lemma 4. *For any Bayesian IC and IR mechanism characterized in Lemma 3, the ex-ante expected revenue is given by*

$$ER = \sum_i \int_v \left\{ \sum_{n=1}^{K} \mu_i^{(n)}(v_i, v_{-i}) \times \underbrace{\left\{ v_i + \delta(n) - \frac{1 - F_i(v_i)}{f_i(v_i)} \right\}}_{MR(v_i, n)} \right\} f(v) dv - \sum_i U_i(\underline{v}) \tag{29}$$

Proof. See Appendix. □

Let $v_{(1)} \geq v_{(2)} \geq \cdots \geq v_{(K)}$ denote the realizations of the highest, the second highest, ..., the Kth highest type among the N bidders. Lemma 4 implies that conditional on any supply level n, the optimal auction always assigns n licenses to the n bidders with highest marginal revenues. It is well known that in Myerson (1981)'s optimal auction, an nth license should be sold if and only if $MR(v_{(n)}, n) = v_{(n)} - \frac{1 - F_i(v_{(n)})}{f_i(v_{(n)})} \geq 0$. However, under the presence of quantity externalities, $MR(v_{(n)}, n) \geq 0$ is a necessary but insufficient condition for total supply to be n. This is because selling each additional license not only creates

[14] $\sum_{n=1}^{K} \mu_i^{(n)}(v_i, v_{-i})$ is the ex-post probability that bidder i wins a license unconditional on supply level given type profile (v_i, v_{-i}).

marginal revenue gain but also creates marginal revenue loss to the auctioneer by reducing marginal revenues extracted from other winning bidders.

Conditional on having allocated $(n-1)$ licenses, the marginal revenue gain to the auctioneer from selling the nth license is the marginal revenue extracted from the nth highest type bidder:

$$MR(v_{(n)}, n) = v_{(n)} + \delta(n) - \frac{1 - F_i(v_{(n)})}{f_i(v_{(n)})} \tag{30}$$

Note that since $MR(v_i, n)$ is strictly increasing in v_i, $MR(v_{(n)}, n)$ is strictly decreasing in n. Comparing Eq. (30) to Eq. (5) shows that $MB(v_{(n)}, n) > MR(v_{(n)}, n)$ for any $v_{(n)}$, for any n. That is, the social marginal benefit from selling an nth license is always greater than the auctioneer's marginal revenue gain from selling an nth license. This result comes from two forces. First, selling an additional license creates positive externalities outside of auction and the auctioneer is unable to extract revenue from these positive externalities. Second, each winning bidder is able to keep some informational rent from the auction, which means that the auctioneer is unable to extract all the surplus created within auction by allocating the licenses. The first force is unique under the presence of quantity externalities, while the second force exists in all private values settings.

Conditional on having allocated $(n-1)$ licenses, the marginal revenue loss to the auctioneer from selling the nth license is the reduction in marginal revenues extracted from all $(n-1)$ highest type bidders who would still win a license when the nth license is not sold:

$$ML(n) = \sum_{k=1}^{n-1} \left[MR(v_{(k)}, n-1) - MR(v_{(k)}, n)\right] = (n-1)[\delta(n-1) - \delta(n)] \tag{31}$$

Note that the marginal revenue loss from selling the nth license is the total negative externalities that selling each additional license imposes on the winning bidders. Comparing Eq. (31) to Eq. (6) shows that $MC(n) = ML(n)$ for any n. That is, the auctioneer's marginal revenue loss from selling an nth license is also the social marginal cost of selling an nth license.

The nth license should be sold in the optimal auction if and only if the marginal revenue gain from selling an nth license is greater than the marginal revenue loss from selling an nth license, i.e., $MR(v_{(n)}, n) \geq ML(n)$. Since the marginal revenue gain $MR(v_{(n)}, n)$ is decreasing in n and the marginal revenue loss $ML(n)$ is increasing in n, the total revenue is maximized at the supply level n^* s.t.

$$\begin{aligned} &MR\left(v_{(n^*)}, n^*\right) \geq ML\left(n^*\right) \\ &MR\left(v_{(n^*+1)}, n^*+1\right) < ML\left(n^*+1\right) \end{aligned} \tag{32}$$

Therefore, the optimal direct mechanism can be constructed as follows. Each bidder i is asked to report its private type v_i. Given any profile of reported types $\hat{v}$, the auctioneer ranks the reported types indescending order: $\hat{v}_{(1)} \geq \hat{v}_{(2)} \geq \cdots \geq \hat{v}_{(N)}$ and allocates licenses according to the following procedure:

(R1) Allocate one license to the bidder with the highest reported type $\hat{v}_{(1)}$ if $MR(v_{(1)}, 1) \geq ML(1)$ and continue to (R2). End the algorithm and sell zero license otherwise.

(R2) Allocate one license to the bidder with the second highest reported type $\hat{v}_{(2)}$ if $MR(v_{(2)}, 2) \geq ML(2)$ and continue to (R3). End the algorithm and sell one license otherwise.

...

(RK) Allocate one license to the bidder with the Kth highest reported type $\hat{v}_{(K)}$ if $MR(v_{(K)}, K) \geq ML(K)$. End the algorithm and sell $(K-1)$ licenses otherwise.

The following proposition gives the formal definition of the optimal direct mechanism:

Proposition 4. *Given any type profile (v_i, v_{-i}), let $V^n(v_i, v_{-i})$ denote the nth highest value among (v_i, v_{-i}). Consider the following mechanism (μ^*, t^*):*

For all $n \in \{1, 2, \cdots, K\}$,[15]

$$\mu_i^{*(n)}(v_i, v_{-i}) = \begin{cases} 1 & \text{if} \quad v_i \geq V^n(v_i, v_{-i}), \\ & \quad MR\left(V^n(v_i, v_{-i}), n\right) \geq ML(n), \\ & \quad MR\left(V^{n+1}(v_i, v_{-i}), n+1\right) < ML(n+1) \\ 0 & \text{else} \end{cases} \tag{33}$$

Any tie is broken randomly. The payment rule is given by

$$t_i^*(v_i, v_{-i}) = \sum_{n=1}^{K} \mu_i^{*(n)}(v_i, v_{-i})\left[v_i + \delta(n)\right] - \int_{\underline{v}}^{v_i} \sum_{n=1}^{K} \mu_i^{*(n)}(s, v_{-i})ds \tag{34}$$

(μ^*, t^*) is an optimal auction among all Bayesian IC and IR mechanisms.

Let v_n^* denote the optimal reserve type that justifies selling n licenses for any $n \in \{1, 2, \cdots, K\}$. The characterization of optimal reserve types follows from Proposition 4.

[15] For $n = K$, $\mu_i^{*(K)}(v_i, v_{-i}) = 1$ if $v_i \geq V^K(v_i, v_{-i})$ and $MR\left(V^K(c_i, c_{-i}), K\right) \geq ML(K)$.

Corollary 3. *The optimal reserve types* $(v_1^*, v_2^*, \cdots, v_K^*)$ *in which* v_n^* *specifies the minimum value that the nth highest type* $v_{(n)}$ *can take for an nth license to be sold in the optimal auction are characterized below:*

$$MR(v_n^*, n) = ML(n) \tag{35}$$

More specifically,

$$v_n^* + \delta(n) - \frac{1 - F_i(v_n^*)}{f_i(v_n^*)} = (n-1)[\delta(n-1) - \delta(n)], \quad \text{for all} \quad n \tag{36}$$

For every unit n, the optimal reserve type v_n^* that justifies selling the nth license to the nth highest type bidder is the threshold type that satisfies $MR(v_n^*, n) = ML(n)$. Since the allocation of an nth license reduces marginal revenues extracted from the $(n-1)$ highest type bidders who would still win a license when the nth license is not allocated, whether it worths to allocate the nth license depends on whether the marginal revenue extracted from the marginal bidder is greater than the loss inrevenues extracted from the $(n-1)$ highest type bidders. For any $v_i \geq v_n^*$, $MR(v_i, n) \geq ML(n)$, implying that the marginal revenue gain from selling an additional license to a marginal bidder with type $v_i \geq v_n^*$ is greater than the marginal revenue loss from the negative externalities imposed on the other $(n-1)$ winners. For any $v_i < v_n^*$, $MR(v_i, n) < ML(n)$, implying that it is not optimal to allocate an nth license to a marginal bidder with type $v_i < v_n^*$.

Furthermore, note that $MR(v_i, n) > MR(v_i, n+1)$ and $ML(n) < ML(n+1)$ for any n, at any v_i, which implies that the optimal reserve types must satisfy $v_1^* < v_2^* < \cdots < v_K^*$. Therefore, in any optimal auction that adopts a sequence of optimal reserve prices $(r_1^*, r_2^*, \cdots, r_K^*)$ where r_n^* specifies the minimum acceptable bid at supply level n, the optimal reserve prices must also satisfy $r_1^* < r_2^* < \cdots < r_K^*$. Given the equilibria characterized in Propositions 2 and 3, it follows that the multi-dimensional uniform-price auction and the Walrasian clock auction can yield the optimal revenue under a sequence of optimal reserve prices that correspond to the optimal reserve types characterized in Corollary 3.

Corollary 4. *The multi-dimensional uniform-price auction and the Walrasian clock auction can implement the optimal revenue in dominant strategy equilibria with a sequence of optimal conditional reserve prices* $(r_1^*, r_2^*, \cdots, r_K^*)$ *given below:*

$$r_n^* = v_n^* + \delta(n), \ \forall n \in \{1, 2, \cdots, K\} \tag{37}$$

in which v_n^* *is characterized by* $MR(v_n^*, n) = ML(n)$.

Note that setting the optimal reserve price schedule requires the auctioneer to know the distribution of bidders' private types $F(.)$. This informational assumption is also required in Myerson (1981). The presence of quantity externalities does not introduce additional informational requirement to the auctioneer in the optimal auction.

Next, compare the optimal reserve type v_n^* to the efficient reserve type v_n^e. For any supply level n, the optimal reserve type v_n^* and the efficient reserve type v_n^e are given by

$$\begin{aligned} MR(v_n^*, n) &= ML(n) \\ MB(v_n^e, n) &= MC(n) \end{aligned} \tag{38}$$

Since $MR(v_i, n) < MB(v_i, n)$ and $ML(n) = MC(n)$ for any n, at any v_i, and both $MR(v_i, n)$ and $MB(v_i, n)$ are strictly increasing in v_i, we must have $v_n^* > v_n^e$ for any n. Given $r_n^* = v_n^* + \delta(n)$ and $r_n^e = v_n^e + \delta(n)$, it follows that $r_n^* > r_n^e$ for any n.

Corollary 5. *For any supply level* $n \in \{1, 2, \cdots, K\}$, *given any type profile, the optimal reserve price* r_n^* *is strictly higher than the efficient reserve price* r_n^e.

Corollary 5 implies that the optimal auction always sells fewer licenses than the efficient auction in expecation given any type profile.

6. Conclusions

This paper constructs both static and dynamic license auctions for selling entry permits to a post-auction market. In such license auction settings, two types of quantity externalities present. First, selling more licenses allows more bidders to enter the post-auction market while reducing the payoff of every individual bidder who enters the market. Second, selling more licenses increases surplus of participants on the other side of the post-auction market. Under the presence of quantity externalities, standard auctions such as uniform-price auctions and ascending clock auctions no longer implements efficiency, and alternative auction formats should be considered. I show that the socially optimal allocation can still be implemented through auctions, and any efficient auction that maximizes social surplus should determine total supply endogenously based on bid profile. More specifically, I propose two practical auctions that choose supply endogenously by allowing the auctioneer to condition reserve prices on supply and allowing the bidders to condition bids on supply. The two proposed auctions are outcome equivalent to the VCG mechanism under efficient reserve prices. Moreover, both auctions can implement the optimal revenue under optimal reserve prices. Both auctions also have simpler design and take shorter time to conclude compared to the Jumping English auction proposed by Gebhardt and Wambach (2008) for allocating licenses under the same setting.

One implication of this paper is that when bidders' values depend on final supply, any efficient auction should allow the auctioneer to condition reserve prices on supply and allow bidders to condition bids on supply. Both auctions proposed in this paper as well as the Jumping English auction proposed by Gebhardt and Wambach (2008) share this feature. Simply introducing a sequence of conditional reserve price into uniform-price auctions or ascending clock auctions with single-dimensional bids can lead to inefficient outcomes, as the expected supply conditional on winning is differentiated across bidders, causing lower type bidders to bid more aggressively than higher type bidders. Simply allowing bidders to condition bids on supply with a single reserve price can also lead to inefficient outcomes, as the social marginal benefit and marginal cost of selling each additional license differ across supply levels, and it is impossible to select the socially optimal supply level with a single reserve price. This paper shows that by allowing the auctioneer to condition reserve prices on supply and allowing bidders to condition bids on supply, the VCG outcome can be practically implemented in both static and dynamic auctions under the presence of quantity externalities.

Another implication of this paper is that auctions can be used as regulation devices in markets where existence of entry-related externalities leads to market failure in firm entry and product selection. This market failure is pointed out in earlier theoretical literature (Spence, 1976; Koenker and Perry, 1981) and still exists in many real markets. This paper shows that, with correctly designed auction rules and reserve prices, any market failure associated with positive or negative externalities induced by firm entry can be eliminated by allocating entry permits in a centralized auction mechanism. Therefore, this paper not only has implications in the field of mechanism design, but also has important policy implications.

In fact, the idea of using market design to address market failures under externalities is not new. For example, the US government assigns tradable emission permits to firms in order to control air pollution.[16] However, most of the tools used to design markets in previous applications are restricted to regulation tools such as quotas, taxes or subsidies, instead of well-designed allocation mechanisms that assign permits to those with the highest values. On the other hand, the potential of designing a centralized allocation mechanism to address market failure under externalities is emerging. The adoption of auctions for allocating car license plates in Shanghai is a pliot example in this space, and there is empirical evidence showing that assigning car license plates through a lottery-based quota system instead of an auction can lead to significant welfare loss due to misallocation of licenses when car buyers have different willingness of pay for licenses (Li, 2018). This paper sheds some light on the application of market design in entry regulations by offering a new perspective for addressing market failures under entry-related externalities.

There are a few future research directions related to this paper. First, this paper assumes that the magnitude of externalities imposed on winning bidders and auction outsiders from selling an additional license only depends on the number of licenses already allocated and does not depend on identity or type of the bidder who wins the marginal license. This can be a practical assumption when bidders are roughly homogeneous in the post-auction market. For example, in car license plate auctions, the externalities that each additional car brings to the society can be viewed as homogeneous if car buyers are equally likely to drive upon getting license plates. However, this assumption may not hold in auctions where bidders have heterogeneous impact in the post-auction market. For example, if car buyers with higher values for licenses are more likely to drive, the externalities associated with the entry of a high-value car buyer can be different from that associated with the entry of a low-value car buyer.[17] In government-sponsored auctions for selling production rights, the externalities associated with the entry of a dominant firm are likely to be different from that associated with the entry of a small firm. One future research direction is to explore the design of efficient auctions to implement the socially optimal outcome when entry-related externalities not only depends on the number of licenses already sold, but also depends on the types or identities of winning bidders.

Second, this paper provides some implications for empirical research. One future research direction is to estimate how bidder valuations change with total supply in auctions and quantify the policy impact of varying supply level on welfare and revenue outcomes.

Third, this paper assumes that all licenses sold in the auction are identical. In some real-world settings, licenses sold in the auction can be heterogeneous. For example, in spectrum auctions, spectrum bands can differ in band width, frequency range, and geographic coverage. In sponsored advertising auctions, advertising slots can differ in display region and rank. A natural extension of the model is to introduce heterogeneity into licenses and explore the design of efficient auctions for allocating heterogeneous items under quantity externalities.

Fourth, this paper assumes that all licenses are sold in the same auction. In some real-world settings, licenses can be sold sequentialy one at a time. Another future research direction is to explore the design of efficient mechanisms when licenses are sold sequentially under the presence of quantity externalities.

Appendix A

Proof of Proposition 1:

[16] According to Canon et al. (2013), the criteria for assigning permits was ad hoc, with the endowment of each polluting unit depending on each firm's fuel input share. Therefore, the mechanism used to assign emission rights is quite different from an auction mechanism.

[17] Li (2018) empirically quantifies the welfare outcomes of two car license plate allocation mechanisms when the negative externalities associated with car usage is increasing in car buyer's willingness to pay for a license plate. Under this type-dependent externalities, the social benefit from assigning a license can be decreasing in the car buyer's private value. This implies that any auction that assigns licenses to those with highest values without considering this type-dependent externalities is inefficient.

Proof. For any bidder i, for any $n \in \{1, 2, \cdots, K\}$, given any profile of bidder i's opponents' reported types $\hat{v}_{-i}$, define $\max(n)\{\hat{v}_{-i}\}$ as the nth highest value among $\hat{v}_{-i}$. Define

$$\hat{V}_n(\hat{v}_{-i}) = \max\left\{\max(n)\{\hat{v}_{-i}\}, v_n^e\right\} \tag{39}$$

in which $v_n^e = (n-1)[\delta(n-1) - \delta(n)] - [\pi^o(n) - \pi^o(n-1)] - \delta(n)$ is the efficient reserve type that justifies selling an nth license as defined in Lemma 1.

For any bidder i, for every feasible supply level $n \in \{1, 2, \cdots, K\}$, given any opponents' reported type profile $\hat{v}_{-i}$, define Scenario n to be the event that $\hat{v}_{-i}$ satisfies the following conditions[18]:

$$\begin{aligned} &MB\Big(\max(n-1)\{\hat{v}_{-i}\}, n\Big) \geq MC(n) \\ &MB\Big(\max(n)\{\hat{v}_{-i}\}, n+1\Big) < MC(n+1) \end{aligned} \tag{40}$$

Note that the scenarios $1, 2, \cdots, K$ partition the space of $\hat{v}_{-i} \in [\underline{v}, \bar{v}]^{N-1}$. That is, for any bidder i, its opponents' type profile $\hat{v}_{-i}$ must fall into one of the K scenarios, in which Scenario n is characterized by conditions (40) for all $n \in \{1, 2, \cdots, K\}$.

For every $n \in \{1, 2, \cdots, K\}$, it is never efficient to sell fewer than $(n-1)$ licenses or more than n licenses given opponents' report $\hat{v}_{-i}$ in Scenario n, regardless of bidder i's report. This is because that no matter what type that bidder i reports, its opponents' reported type profile $\hat{v}_{-i}$ always justifies allocating an $(n-1)$th license and always fails to justify allocating an $(n+1)$th license in the auction in Scenario n.

Under Scenario n, bidder i wins a license and make final supply equal to n if and only if

$$\hat{v}_i \geq \max\left\{\max(n)\{\hat{v}_{-i}\}, v_n^e\right\} \tag{41}$$

i.e., $\hat{v}_i \geq \hat{V}_n(v_{-i})$. Bidder i does not win a license otherwise.

Bidder i's payoff from winning under Scenario n is given by

$$\begin{aligned} & v_i + \delta(n) - p_n(\hat{v}) \\ = \quad & v_i + \delta(n) - \max\left\{\max(n)\{\hat{v}_{-i}\} + \delta(n),\ v_n^e + \delta(n)\right\} \\ = \quad & \min\left\{v_i - \max(n)\{\hat{v}_{-i}\},\ v_i - v_n^e\right\} \\ = \quad & v_i - \max\left\{\max(n)\{\hat{v}_{-i}\}, v_n^e\right\} \\ = \quad & v_i - \hat{V}_n(v_{-i}) \end{aligned} \tag{42}$$

Therefore, bidder i's payoff from winning when opponents' reports $\hat{v}_{-i}$ satisfy Scenario n is non-negative if and only if $v_i \geq \hat{V}_n(\hat{v}_{-i})$. Since bidder i wins if and only if $\hat{v}_i \geq \hat{V}_n(v_{-i})$, reporting $\hat{v}_i = v_i$ maximizes bidder i's payoff when the opponents' reports $\hat{v}_{-i}$ satisfy Scenario n, for every $n \in \{1, 2, \cdots, K\}$. Since Scenarios $1, 2, \cdots, K$ partition the space of $\hat{v}_{-i}$, reporting $\hat{v}_i = v_i$ maximizes bidder i's payoff given any opponents' report $\hat{v}_{-i}$, which means the truth-telling is a dominant strategy equilibrium in the VCG mechanism. □

Proof of Proposition 2:

Proof. For any bidder i, for all $n \in \{1, 2, \cdots, K\}$, let $\Pi_i^n\big(b_i, b_{-i}\big)$ denote bidder i's ex-post payoff from winning a license when total supply equals to n, given bidder i's bid $b_i \in \mathbb{R}^K$ and opponents' strategy $b_{-i} \in \mathbb{R}^{K(N-1)}$. The second price payment rule implies that $\Pi_i^n(b_i, b_{-i})$ depends only on b_{-i}^n and does not depend on bidder i's own bid b_i. Let $P_i^n\big(b_i, b_{-i}\big)$ denote the ex-post probability that bidder i wins a license when total supply equals to n given bid profile (b_i, b_{-i}). In a dominant strategy equilibrium, bidder i chooses bid b_i to maximize its ex-post payoff given any opponents' bids b_{-i}:

$$\max_{b_i} \sum_{n=1}^{K} P_i^n\big(b_i, b_{-i}\big) \times \Pi_i^n\big(b_i, b_{-i}\big), \quad \text{for any} \quad b_{-i} \tag{43}$$

Given any opponents' bid profile b_{-i}, let $\hat{b}_{-i}^1$ denote the highest bid conditional on supply equals to 1 among bidder i's opponents. Let $\hat{b}_{-i}^2$ denote the highest bid conditional on supply equals to 2 among bidder i's opponents who do not win in any round before round 2. Similarly, for any $n \in \{3, 4, \cdots, K\}$, let $\hat{b}_{-i}^n$ denote the highest bid conditional on supply equals to n among bidder i's opponents who do not win in any round before round n.

Given any bid profile (b_i, b_{-i}), the ex-post probability for bidder i to win a license when supply equals to 1 is given by

$$P_i^1(b_i, b_{-i}) = \begin{cases} 1 & \text{if} \quad b_i^1 \geq \max\{\hat{b}_{-i}^1, r_1\}, \hat{b}_{-i}^2 < r_2 \\ 0 & \text{otherwise} \end{cases} \tag{44}$$

[18] Since $MB(v_i, 1) > MC(1)$ for any v_i, the first condition is automatically satisfied in scenario 1. Scenario 1 can be characterized by $MB(\max(1)\{\hat{v}_{-i}\}, 2) < MC(2)$ only. Similarly, Scenario K can be characterized by $MB\big(\max(K-1)\{\hat{v}_{-i}\}, K\big) \geq MC(K)$ only.

That is, bidder i wins a license when supply is set to 1 if and only if b_i^1 is the greatest bid at supply n and passes the reserve price r_1, **and** the highest bid among bidder i's opponents who have not won in round 2 fails to meet the reserve price r_2.

Given any bid profile (b_i, b_{-i}), the ex-post probability for bidder i to win a license when supply equals to 2 is given by

$$P_i^2(b_i, b_{-i}) = \begin{cases} 1 \quad \text{if} & b_i^1 \geq \max\{\hat{b}^1_{-i}, r_1\} \\ & \hat{b}^2_{-i} \geq r_2, \hat{b}^3_{-i} < r_3 \\ \quad\ \text{or} & b_i^1 < \hat{b}^1_{-i}, b_i^2 \geq \max\{\hat{b}^2_{-i}, r_2\} \\ & \hat{b}^1_{-i} \geq r_1, \hat{b}^3_{-i} < r_3 \\ 0 \quad \text{otherwise} & \end{cases} \tag{45}$$

Note that bidder i can win a license when supply is set at 2 under 2 potential scenarios. In the first scenario, bidder i wins a license in round (R1) by bidding $b_i^1 \geq \max\{\hat{b}^1_{-i}, r_1\}$, the highest bidder among i's opponents wins a license in round (R2) by bidding $\hat{b}^2_{-i} \geq r_2$. In the second scenario, bidder i loses to an opponent who bids $\hat{b}^1_{-i} \geq r_1$ in (R1), but wins a license in (R2) by bidding $b_i^2 \geq \max\{\hat{b}^2_{-i}, r_2\}$. In both scenarios, the highest bidder in the remaining opponents in (R3) fails to meet the reserve price r_3, so only two licenses are sold. Under the same reasoning, there are n scenarios for bidder i to win when supply equals to n (basically winning in round 1, 2, $\cdots$, n when supply equals to n).

Eqs. (44) and (45) show that b_i^2 only enters $P_i^2(b_i, b_{-i})$ and does not enter $P_i^1(b_i, b_{-i})$, while b_i^1 enters both $P_i^1(b_i, b_{-i})$ and $P_i^2(b_i, b_{-i})$. Moreover, given any b_{-i}, increasing b_i^1 while keeping b_i^2 constant increases $P_i^2(b_i, b_{-i})$ only if $b_i^2 < \max\{\hat{b}^2_{-i}, r_2\}$. Decreasing b_i^1 while keeping b_i^2 constant does not affect $P_i^2(b_i, b_{-i})$.

For all $n \in \{1, 2, \cdots, K\}$, the probability for bidder i to win a license when supply equals to n is given by[19]

$$P_i^n(b_i, b_{-i}) = \begin{cases} 1 \quad \text{if} & b_i^1 \geq \max\{\hat{b}^1_{-i}, r_1\} \\ & \hat{b}^k_{-i} \geq r_k,\ \forall k \in \{2, 3, \cdots, n\}, \hat{b}^{n+1}_{-i} < r_{n+1} \\ \quad\ \text{or} & b_i^1 < \hat{b}^1_{-i}, b_i^2 \geq \max\{\hat{b}^2_{-i}, r_2\} \\ & \hat{b}^k_{-i} \geq r_k,\ \forall k \in \{1, 3, \cdots, n\}, \hat{b}^{n+1}_{-i} < r_{n+1} \\ & \cdots \\ \quad\ \text{or} & b_i^k < \hat{b}^k_{-i},\ \forall k \in \{1, 2, \cdots, n-1\}, b_i^n \geq \max\{\hat{b}^n_{-i}, r_n\}, \\ & \hat{b}^k_{-i} \geq r_k,\ \forall k \in \{1, 2, \cdots, n-1\}, \hat{b}^{n+1}_{-i} < r_{n+1} \\ 0 \quad \text{otherwise} & \end{cases} \tag{46}$$

Eq. (46) shows that for all n, given any b_{-i}, $P_i^n(b_i, b_{-i})$ depends only on b_i^k for $k \leq n$ and b_{-i} but does not depend on any b_i^m for $m > n$. That is, for any supply level k, each bidder i's bid b_i^k only enters $P_i^n(b_i, b_{-i})$ for $n \geq k$, ex-post probabilities of winning conditional on supply level being greater than or equal to k, but does not affect any $P_i^n(b_i, b_{-i})$ for $n < k$, ex-post probabilities of winning conditional on supply level being lower than k. Moreover, for any $n > k$, holding all b_i^j for $j \neq k$ and b_{-i} constant, increasing b_i^k increases $P_i^n(b_i, b_{-i})$ only if $b_i^n < \max\{\hat{b}^n_{-i}, r_n\}$, while decreasing b_i^k does not affect $P_i^n(b_i, b_{-i})$.

I next formally prove that bidding $b_i^n = v_i + \delta(n)$ when $v_i + \delta(n) \geq r_n$ and bidding $b_i^n = 0$ when $v_i + \delta(n) < r_n$ for all n is a dominant strategy equilibrium in the multi-dimensional uniform-price auction.

First, consider bidder i's bid conditional on the highest possible supply level $n = K$. Since b_i^K only enters $P_i^K(b_i, b_{-i})$ and does not enter any $P_i^n(b_i, b_{-i})$ or $\Pi_i^n(b_i, b_{-i})$ for $n < K$, a dominant strategy b_i^{K*} should solve

$$\max_{b_i^K} P_i^K(b_i, b_{-i}) \times \Pi_i^K(b_i, b_{-i}), \quad \forall b_{-i} \tag{47}$$

Given the second-price payment rule and reserve price r_K, it is a dominant strategy to bid $b_i^K = v_i + \delta(n)$ if $v_i + \delta(n) \geq r_K$ and bid $b_i^K = 0$ otherwise.

Next, consider bidder i's bid conditional on supply level $K-1$. b_i^{K-1} enters both $P_i^{K-1}(b_i, b_{-i})$ and $P_i^K(b_i, b_{-i})$. Therefore, a dominant strategy b_i^{K-1} should solve

$$\max_{b_i^{K-1}} P_i^{K-1}(b_i, b_{-i}) \times \Pi_i^{K-1}(b_i, b_{-i}) + P_i^K(b_i, b_{-i}) \times \Pi_i^K(b_i, b_{-i}) \quad \forall b_{-i} \tag{48}$$

Given the second price payment rule and r_{K-1}, the objective function (48) is maximized by bidding $b_i^{K-1} = v_i + \delta(n)$ when $v_i + \delta(n) \geq r_{K-1}$ and bidding $b_i^{K-1} = 0$ otherwise. The reasoning is given below:

- Suppose $v_i + \delta(K-1) \geq r_{K-1}$. Consider an alternative strategy of bidding $b_i^{K-1} < v_i + \delta(K-1)$. It decreases $P_i^{K-1}(b_i, b_{-i})$ only if $v_i + \delta(K-1) > \hat{b}^{K-1}_{-i}$, i.e., $\Pi_i^{K-1}(b_i, b_{-i}) > 0$. It does not affect $P_i^K(b_i, b_{-i})$. Therefore, for any b_{-i}, bidding $b_i^{K-1} = v_i + \delta(K-1)$ when $v_i + \delta(K-1) \geq r_{K-1}$ yields a weakly better payoff than bidding $b_i^{K-1} < v_i + \delta(K-1)$.

[19] For $n = K$, define $\hat{b}^{K+1}_{-i} = 0$ and $r_{K+1} = \infty$ so that $\hat{b}^{K+1}_{-i} < r_{K+1}$ always hold.

• Suppose $v_i + \delta(K-1) \geq r_{K-1}$. Consider an alternative strategy of bidding $b_i^{K-1} > v_i + \delta(K-1)$. It increases $P_i^{K-1}(b_i, b_{-i})$ only if $\Pi_i^{K-1}(b_i, b_{-i}) < 0$. It increases $P_i^K(b_i, b_{-i})$ only if $b_i^K = v_i + \delta(K) < \max\{\hat{b}_{-i}^K, r_K\}$, i.e., $\Pi_i^K(b_i, b_{-i}) < 0$. Therefore, for any b_{-i}, bidding $b_i^{K-1} = v_i + \delta(K-1)$ when $v_i + \delta(K-1) \geq r_{K-1}$ yields a weakly better payoff than bidding $b_i^{K-1} > v_i + \delta(K-1)$.

• Suppose $v_i + \delta(K-1) < r_{K-1}$. Consider an alternative strategy of bidding $b_i^{K-1} \geq r_{K-1}$. It increases $P_i^{K-1}(b_i, b_{-i})$ when $\Pi_i^{K-1}(b_i, b_{-i}) < 0$. It increases $P_i^K(b_i, b_{-i})$ only if $b_i^K = v_i + \delta(K) < \max\{\hat{b}_{-i}^K, r_K\}$, i.e., $\Pi_i^K(b_i, b_{-i}) < 0$. Any alternative strategy of bidding $b_i^{K-1} < r_n$ yields the same payoff as bidding $b_i^{K-1} = 0$. Therefore, for any b_{-i}, bidding $b_i^{K-1} = 0$ when $v_i + \delta(K-1) < r_{K-1}$ yields a weakly better payoff than bidding $b_i^{K-1} > 0$.

Using a similar reasoning, it can be shown that for all $n \in \{1, 2, \cdots, K\}$, it is a dominant strategy to bid $b_i^n = v_i + \delta(n)$ if $v_i + \delta(n) \geq r_n$ and bid $b_i^n = 0$ otherwise.

Note that given b_i^k does not enter $P_i^n(b_i, b_{-i})$ for any $n < k$, it is impossible to manipulate probability of winning $P_i^n(b_i, b_{-i})$ through b_i^k for any $n < k$. Although it is possible to increase probability of winning $P_i^n(b_i, b_{-i})$ through increasing b_i^k for $n > k$, any such manipulation makes i weakly worse off. Therefore, each bidder i simply chooses b_i^k to maximize the expected payoff conditional on supply being equal to k in equilibrium. The multi-dimensional uniform-price auction can be viewed as a set of simultaneous independent second-price auctions conditional on supply being equal to 1, 2, ⋯, K, respectively. It follows that truth-telling is a dominant strategy equilibrium conditional on every supply level. □

Proof of Proposition 3:

Proof. For all $n \in \{K, K-1, \cdots, 1\}$, first consider each active bidder's drop-out strategy in the ascending clock auction in the final round (Rn) where $k(r_n) \geq n$. When $k(r_n) = n$, the market clearing price is r_n and the auction ends immediately, so there is no need to discuss bidders' strategies. When $k(r_n) > n$, the auction is equivalent to an ascending clock auction with fixed supply of n identical items and $k(r_n)$ bidders with single-unit demands. It is a dominant strategy for each bidder to drop out at its true value $v_i + \delta(n)$ at supply n.

Next, consider each bidder's strategy at the beginning for each round (Rn), when the clock price is set to r_n and supply is set to n. Each bidder i needs to decide whether to state "in" or "out" at r_n. For each bidder i, participating in round (Rn) implies that it pays at least r_n conditional on winning in round (Rn). Consider the strategy for any bidder i with type $v_i < r_n - \delta(n)$:

- If there are fewer than $(n-1)$ opponents stating "in" at clock price r_n, then stating "in" and "out" yields the same zero payoff for bidder i in round (Rn), since no allocation will occur in (Rn) and the auction will proceed to round (Rn-1) under either strategy.
- If there are exactly $(n-1)$ opponents stating "in" at clock price r_n, then bidder i wins a license with a negative payoff of $v_i + \delta(n) - r_n < 0$ by stating "in" and gets payoff of zero by stating "out" in round (Rn).
- If there are more than $(n-1)$ opponents stating "in" at clock price r_n, then bidder i gets a non-positive payoff by stating "in" and gets a zero payoff by stating "out" in (Rn), since the auction will end in (Rn) with a price strictly higher than r_n.

Therefore, at the beginning of every round (Rn), it is a weakly dominant strategy to state "out" at initial clock price r_n for any bidder with type $v_i < r_n - \delta(n)$.

Next, consider the strategy for any bidder with type $v_i \geq r_n - \delta(n)$:

- If there are fewer than $(n-1)$ opponents stating "in" at the price of r_n, then stating "in" and "out" yields the same zero payoff for bidder i in round (Rn), since no allocation will occur in (Rn) and the auction will proceed to round (Rn-1) under either strategy.
- If there are exactly $(n-1)$ opponents stating "in" at the price of r_n, that implies there are $(n-1)$ opponents with types $v_j \geq r_n - \delta(n)$, since no bidder with type $v_j < r_n - \delta(n)$ would state "in" as proved above.
  - If bidder i state "in" at r_n, the auction ends immediately and bidder i gets a payoff of $v_i + \delta(n) - r_n \geq 0$.
  - If bidder i state "out" at r_n, the auction will proceed to round (Rn-1). Bidder i will get a payoff of zero if it does not participate in (Rn-1). Suppose bidder i participate in (Rn-1). Since all bidders who state "in" at r_n are required to remain in the auction when price jumps down to r_{n-1}, all of these $(n-1)$ active bidders in round (Rn) must remain active in round (Rn-1) and the auction will end in (Rn-1) with a probability of 1. Let v_{n-1} denote the lowest type among those $(n-1)$ active bidders in (Rn), then $v_{n-1} \geq r_n - \delta(n)$. This bidder's drop out price will be $v_{n-1} + \delta(n-1)$ in the ascending clock auction in round (Rn-1).
    - If bidder i drops out before clock price reaches $v_{n-1} + \delta(n-1)$, then bidder i does not get a license and gets zero payoff.
    - If bidder i stays in the auction when the clock price reaches $v_{n-1} + \delta(n-1)$ and wins one out of $(n-1)$ licenses, then bidder i gets a payoff of $v_i - v_{n-1}$. However, since $v_{n-1} \geq r_n - \delta(n)$, it follows that $v_i - v_{n-1} \leq v_i + \delta(n) - r_n$, i.e., even if bidder i wins in round (Rn-1) after stating "out" at the beginning of (Rn), its payoff will be weakly lower than its payoff from stating "in" and winning in (Rn) when there are $(n-1)$ opponents stating "in" at the beginning of (Rn).

  Therefore, bidder i is always worse off by stating "out" instead of "in" under every possible scenarios when there are exactly $(n-1)$ opponents stating "in" at price r_n.

- If there are more than $(n-1)$ opponents stating “in” at clock price r_n, then bidder i gets a non-negative payoff by stating “in” at r_n and gets a zero payoff by stating “out” at r_n.

Therefore, for any bidder with type $v_i \geq r_n - \delta(n)$, it is a dominant strategy to state “in” at clock price r_n at the beginning of every round (Rn). □

Proof of Lemma 3[20]:

Proof. I first show that any Bayesian IC and IR mechanism must satisfy the characterizations in Lemma 3, then I show that any mechanism satisfying the characterizations in Lemma 3 must be Bayesian incentive compatible and individually rational.

Suppose (μ, t) is a Bayeisan IC and IR mechanism. Let $\pi(v_i, n) = v_i + \delta(n)$. According to the Bayesian IC and IR condition, for any bidder i, for any true type v_i and any reported type v_i',

$$\begin{aligned} U_i(v_i) \geq\ & \int_{v_{-i}} \left\{ \sum_{n=1}^K \pi(v_i, n)\mu_i^{(n)}(v_i', v_{-i}) - t_i(v_i', v_{-i}) \right\} f_{-i}(v_{-i})dv_{-i} \\ \geq\ & \int_{v_{-i}} \left\{ \sum_{n=1}^K \left[\pi(v_i', n) + (v_i - v_i')\right]\mu_i^{(n)}(v_i', v_{-i}) - t_i(v_i', v_{-i}) \right\} f_{-i}(v_{-i})dv_{-i} \\ \geq\ & \int_{v_{-i}} \left\{ \sum_{n=1}^K \pi(v_i', n)\mu_i^{(n)}(v_i', v_{-i}) - t_i(v_i', v_{-i}) \right\} f_{-i}(v_{-i})dv_{-i} \\ & + \int_{v_{-i}} \left\{ \sum_{n=1}^K \left(v_i - v_i'\right)\mu_i^{(n)}(v_i', v_{-i}) \right\} f_{-i}(v_{-i})dv_{-i} \\ \geq\ & U_i(v_i') + (v_i - v_i') \int_{v_{-i}} \sum_{n=1}^K \mu_i^{(n)}(v_i', v_{-i}) f_{-i}(v_{-i})dv_{-i} \end{aligned} \tag{49}$$

Therefore,

$$\begin{aligned} U_i(v_i) &\geq U_i(v_i') + (v_i - v_i') \int_{v_{-i}} \sum_{n=1}^K \mu_i^{(n)}(v_i', v_{-i}) f_{-i}(v_{-i})dv_{-i} \\ U_i(v_i') &\geq U_i(v_i) + (v_i' - v_i) \int_{v_{-i}} \sum_{n=1}^K \mu_i^{(n)}(v_i, v_{-i}) f_{-i}(v_{-i})dv_{-i} \end{aligned} \tag{50}$$

which can be rewritten as

$$\begin{aligned} U_i(v_i) - U_i(v_i') &\geq (v_i - v_i') \int_{v_{-i}} \sum_{n=1}^K \mu_i^{(n)}(v_i', v_{-i}) f_{-i}(v_{-i})dv_{-i} \\ U_i(v_i) - U_i(v_i') &\leq (v_i - v_i') \int_{v_{-i}} \sum_{n=1}^K \mu_i^{(n)}(v_i, v_{-i}) f_{-i}(v_{-i})dv_{-i} \end{aligned} \tag{51}$$

The inequalities imply that if $v_i' < v_i$, then $U_i(v_i) - U_i(v_i') > 0$.

Suppose $v_i' < v_i$. Divide both sides by $(v_i - v_i')$ and take limit:

$$U_i'(v_i) = \lim_{v_i' \to v_i} \frac{U_i(v_i) - U_i(v_i')}{v_i - v_i'} = \int_{v_{-i}} \sum_{n=1}^K \mu_i^{(n)}(v_i, v_{-i}) f_{-i}(v_{-i})dv_{-i} > 0 \tag{52}$$

Therefore, the Bayesian IC condition implies

$$U_i(v_i) = U_i(\underline{v}) + \int_{\underline{v}}^{v_i} \int_{v_{-i}} \sum_{n=1}^K \mu_i^{(n)}(s, v_{-i}) f_{-i}(v_{-i})dv_{-i}ds \tag{53}$$

Therefore, for all $v_i' \leq v_i$,

$$\int_{v_{-i}} \sum_{n=1}^K \mu_i^{(n)}(v_i', v_{-i}) f_{-i}(v_{-i})dv_{-i} \leq \int_{v_{-i}} \sum_{n=1}^K \mu_i^{(n)}(v_i, v_{-i}) f_{-i}(v_{-i})dv_{-i} \tag{54}$$

Since $U_i'(v_i) > 0$, the Bayesian IR condition $U_i(v_i) \geq 0$ for all $v_i \in [\underline{v}, \bar{v}]$ implies

$$U_i(\underline{v}) \geq 0 \tag{55}$$

Therefore, any Bayesian IC and IR mechanism must satisfy the three conditions (53), (54), and (55) characterized in Lemma 3. It left to show that any mechanism (μ, t) that satisfies the characterization in Lemma 3 must be Bayesian IC and IR.

Eq. (53) and inequality (54) implies that $U_i(v_i) \geq U_i(\underline{v})$ for all $v_i \geq \underline{v}$. $U_i(\underline{v}) \geq 0$ implies Bayesian IR.

Suppose $v_i' \leq v_i$, then

$$\begin{aligned} U_i(v_i) =\ & U_i(v_i') + \int_{v_i'}^{v_i} \left[\int_{v_{-i}} \sum_{n=1}^K \mu_i^{(n)}(s, v_{-i}) f_{-i}(v_{-i})dv_{-i} \right] ds \\ \geq\ & U_i(v_i') + \int_{v_i'}^{v_i} \left[\int_{v_{-i}} \sum_{n=1}^K \mu_i^{(n)}(v_i', v_{-i}) f_{-i}(v_{-i})dv_{-i} \right] ds \\ =\ & U_i(v_i') + (v_i - v_i') \int_{v_{-i}} \sum_{n=1}^K \mu_i^{(n)}(v_i', v_{-i}) f_{-i}(v_{-i})dv_{-i} \end{aligned} \tag{56}$$

According to inequality (50), this condition implies Bayesian IC. □

[20] The proof of Lemma 3 follows from Myerson (1981).

Proof of Lemma 4:

Proof. For each bidder i, the ex-ante expected payoff is given below:

$$\begin{aligned} E_{v_i}\left[U_i(v_i)\right] &= U_i(\underline{v}) + \int_{v_{-i}} \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^{v_i} \sum_{n=1}^{K} \mu_i^{(n)}(s, v_{-i}) ds f_i(v_i) dv_i f_{-i}(v_{-i}) dv_{-i} \\ &= U_i(\underline{v}) + \int_{v_{-i}} \int_{\underline{v}}^{\bar{v}} \left[\int_s^{\bar{v}} \sum_{n=1}^{K} \mu_i^{(n)}(s, v_{-i}) f_i(v_i) dv_i \right] ds f_{-i}(v_{-i}) dv_{-i} \\ &= U_i(\underline{v}) + \int_{v_{-i}} \int_{\underline{v}}^{\bar{v}} \left(1 - F_i(s)\right) \sum_{n=1}^{K} \mu_i^{(n)}(s, v_{-i}) ds f_{-i}(v_{-i}) dv_{-i} \\ &= U_i(\underline{v}) + \int_{v_{-i}} \int_{\underline{v}}^{\bar{v}} \left(1 - F_i(v_i)\right) \sum_{n=1}^{K} \mu_i^{(n)}(v_i, v_{-i}) dv_i f_{-i}(v_{-i}) dv_{-i} \\ &= U_i(\underline{v}) + \int_{v} \left\{ \frac{1-F_i(v_i)}{f_i(v_i)} \times \sum_{n=1}^{K} \mu_i^{(n)}(v_i, v_{-i}) \right\} f(v) dv \end{aligned} \tag{57}$$

The expected total surplus generated within the auction is given by

$$TS = \sum_i \int_v \left\{ \sum_{n=1}^{K} \mu_i^{(n)}(v_i, v_{-i}) \pi(v_i, n) \right\} f(v) dv \tag{58}$$

The seller's expected revenue can be derived by subtracting the expected total payoff of bidders from the expected total surplus:

$$\begin{aligned} ER = \quad & TS - \sum_i E_{v_i}\left[U_i(v_i)\right] \\ = \quad & \sum_i \int_v \left\{ \sum_{n=1}^{K} \mu_i^{(n)}(v_i, v_{-i}) \pi(v_i, n) \right\} f(v) dv \\ & - \sum_i \left\{ U_i(\underline{v}) + \int_v \left\{ \frac{1-F_i(v_i)}{f_i(v_i)} \times \sum_{n=1}^{K} \mu_i^{(n)}(v_i, v_{-i}) \right\} f(v) dv \right\} \\ & = \sum_i \int_v \left\{ \sum_{n=1}^{K} \mu_i^{(n)}(v_i, v_{-i}) \times \left\{ \pi(v_i, n) - \frac{1-F_i(v_i)}{f_i(v_i)} \right\} \right\} f(v) dv - \sum_i U_i(\underline{v}) \\ & = \sum_i \int_v \left\{ \sum_{n=1}^{K} \mu_i^{(n)}(v_i, v_{-i}) \times \left\{ v_i + \delta(n) - \frac{1-F_i(v_i)}{f_i(v_i)} \right\} \right\} f(v) dv - \sum_i U_i(\underline{v}) \end{aligned} \tag{59}$$

□

Proof of Proposition 4:

Proof. The seller's problem is to maximize

$$\begin{aligned} ER \quad &= \sum_i \int_v \left\{ \sum_{n=1}^{K} \mu_i^{(n)}(v_i, v_{-i}) \times \left\{ v_i + \delta(n) - \frac{1-F_i(v_i)}{f_i(v_i)} \right\} \right\} f(v) dv - \sum_i U_i(\underline{v}) \\ &= \sum_i \int_v \left\{ \sum_{n=1}^{K} \mu_i^{(n)}(v_i, v_{-i}) \times MR(v_i, n) \right\} f(v) dv - \sum_i U_i(\underline{v}) \end{aligned} \tag{60}$$

subject to $U_i(\underline{v}) \geq 0$ and the feasibility constraints characterized in Definition 5. It follows that $U_i(\underline{v}) = 0$ in the optimal auction. For every feasible supply level n, it is optimal to rank bidders according to their marginal revenues $MR(v_i, n)$ and allocate a license to each of the n highest marginal revenue bidders. Since $MR(v_i, n)$ is increasing in v_i, it is equivalent to say that conditional on selling n licenses, it is optimal to allocate one license to each of the n highest type bidders.

The marginal revenue gain from selling the nth license is given by

$$MR(v_{(n)}, n) = v_{(n)} + \delta(n) - \frac{1 - F_i(v_{(n)})}{f_i(v_{(n)})} \tag{61}$$

which is the marginal revenue extracted from the nth highest type bidder. $MR(v_{(n)}, n)$ is decreasing in n.

The marginal revenue loss from selling the nth license is given by

$$ML(n) = (n-1)[\delta(n-1) - \delta(n)] \tag{62}$$

which is the loss in marginal revenues extracted from the $(n-1)$ highest type bidders who would still win a license if supply is restricted to $n-1$. $ML(n)$ is increasing in n.

Therefore, the revenue-maximizing supply level n^* must satisfy

$$\begin{aligned} & MR\left(v_{(n)}, n\right) \geq ML(n) \\ & MR\left(v_{(n+1)}, n+1\right) < ML(n+1) \end{aligned} \tag{63}$$

If follows that the allocation rule in the optimal auction is given by

$$\mu_i^{*(n)}(v_i, v_{-i}) = \begin{cases} 1 & \text{if} \quad v_i \leq V^n(v_i, v_{-i}), \\ & \quad MR\left(V^n(v_i, v_{-i}), n\right) \geq ML(n), \\ & \quad MR\left(V^{n+1}(v_i, v_{-i}), n+1\right) < ML(n) \\ 0 & \text{else} \end{cases} \tag{64}$$

for all $n \in \{1, 2, \cdots, K\}$[21], where $V^n(v_i, v_{-i})$ is the nth highest type given a profile of reported types (v_i, v_{-i}). Any tie is broken randomly.

The IC and IR conditions imply that

$$\begin{aligned} U_i(\underline{v}) &= U_i(v_i) - \int_{\underline{v}}^{v_i} \int_{v_{-i}} \sum_{n=1}^{K} \mu^{(n)}(s, v_{-i}) f_{-i}(v_{-i}) dv_{-i} ds \\ &= \int_{v_{-i}} \left[\sum_{n=1}^{K} \mu^{(n)}(v_i, v_{-i}) \left[v_i + \delta(n)\right] - \int_{\underline{v}}^{v_i} \sum_{n=1}^{K} \mu^{(n)}(s, v_{-i}) ds - t_i(v_i, v_{-i}) \right] f_{-i}(v_{-i}) dv_{-i} \\ &= 0 \end{aligned} \tag{65}$$

It follows that $t_i^*(v_i, v_{-i})$ can be set as

$$t_i^*(v_i, v_{-i}) = \sum_{n=1}^{K} \mu^{(n)}(v_i, v_{-i}) \left[v_i + \delta(n)\right] - \int_{\underline{v}}^{v_i} \sum_{n=1}^{K} \mu^{(n)}(s, v_{-i}) ds \tag{66}$$

(μ^*, t^*) is an optimal mechanism among all Bayesian IC and IR mechanisms. □

References

Canon, C., Friebel, G., Seabright, P., 2013. Market design and market failure. In: The Handbook of Rational Choice Social Research. Stanford Social Sciences, pp. 473–512. Chapter 14.
Dana, J.D., Spier, K.E., 1994. Designing a private industry. J. Public Econ. 53, 127–147.
Gebhardt, G., Wambach, A., 2008. Auctions to implement the efficient market structure. Int. J. Ind. Organ. 26 (3), 846–859.
Ghosh, A., Mahdian, M., 2008. Externalities in Online Advertising, Proceedings of the 17th International Conference on World Wide Web. Beijing, China.
Hansen, R., 1988. Auctions with endogenous quantity. RAND J. Econ. 19 (1), 44–58.
Hummel, P., McAfee, P., 2014. Position auctions with externalities. In: International Conference on Web and Internet Economics, 2014, pp. 417–422.
Izmalkov, S., Khakimova, D., Romanyuk, G., 2016. Position Auctions with Endogenous Supply. Working Paper.
Jehiel, P., Moldovanu, B., 2000. License Auctions and Market Structure. CEPR Discussion Papers 2530, C.E.P.R. Discussion Papers.
Jehiel, P., Moldovanu, B., 2004. The design of an efficient private industry. Journal of the European Economic Association, 2, 516–525. 2-3.
Jehiel, P., Moldovanu, B., Stacchetti, E., 1996. How (not) to sell nuclear weapons. Am. Econ. Rev. 86 (4), 814–829.
Jehiel, P., Moldovanu, B., Stachhetti, E., 1999. Multidimensional mechanism design for auctions with externalities. J. Econ. Theory 85 (2), 258–293.
Katz, M., Shapiro, C., 1986. How to license intangible property. Quart. J. Econ. 101 (3), 567–589.
Koenker, R., Perry, M., 1981. Product differentiation, monopolistic competition, and public policy. Bell J. Econ. 12 (1), 217–231.
Lengwiler, Y., 1999. The multiple unit auction with variable supply. Econ. Theory 14 (2), 373–392.
Li, S., 2018. Better lucky than rich? Welfare analysis of automobile license allocations in Beijing and Shanghai. Rev. Econ. Stud. 85 (4), 2389–2428.
Myerson, R.B., 1981. Optimal auction design. Math. Oper. Res. 6 (1), 58–73.
Ozcan, R., 2004. Sequential Auctions with Endogenously Determined Reserve Prices. Boston College Working Papers in Economics 592, Boston College Department of Economics.
Rodriguez, G., 1997. Auctions of Licenses and Market Structure. Universitat Pompeu Fabra Economics. Working Paper No. 209.
Spence, M., 1976. Production selection, fixed costs, and monopolistic competition. Rev. Econ. Stud. 43 (2), 217–235.
Varma, G., 2002. Standard auctions with identity-dependent externalities. RAND J. Econ. 33 (4), 689–708.

[21] Only conditions $v_i \le V^K(v_i, v_{-i})$ and $MR\big(V^K(v_i, v_{-i}), K\big) \ge ML(K)$ are needed for $n = K$.