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Position auctions with multi-unit demands ☆

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ABSTRACT

This paper studies the design of position auctions when bidders have multi-unit demands for advertising positions. I propose an ascending clock auction with two stages: allocation stage and assignment stage. The allocation stage determines the quantity of positions assigned to each advertiser using a generalized version of the Ausubel (2004) auction under the context of differentiated items. The assignment stage determines the ranking of advertisements using a generalized version of the generalized English auction under the context of multi-unit demands. I show that this two-stage ascending clock auction dynamically implements the VCG outcome in an ex post perfect equilibrium under pure private values.

1. Introduction

Position auctions are used by many search engines such as Google and Yahoo! to allocate sponsored advertising slots to advertisers. In addition to search engines, online social platforms such as Facebook and Twitter, online review and booking platforms such as Yelp, Tripadvisor and Expedia, and online shopping platforms such as Amazon and eBay have all been using auctions to allocate sponsored advertising slots to suppliers and business owners.[2] Position auctions have become a major revenue source for many of these two-sided platforms and have attracted extensive research interests in the past decade.

This paper constructs an efficient position auction that allows bidders to have multi-unit demands. The proposed auction implements the efficient outcome with a single ascending clock in two stages: allocation stage and assignment stage. The allocation stage determines the quantity of advertising slots allocated to each bidder. The assignment stage determines the rankings associated with each bidder's advertising slots. The proposed two-stage ascending clock auction implements the VCG outcome in an ex post perfect equilibrium under pure private values. Compared to the Ausubel (2006) auction for selling heterogeneous commodities using multiple clocks, the proposed auction offers a simpler format to dynamically implement the VCG outcome under the special setting of position auctions.

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[2] Many papers have discussed the use of sponsored advertising auctions in online platforms, including Edelman et al. (2007), Varian (2007), Athey and Ellison (2011), Goldman and Rao (2014), Jeziorski and Segal (2015), etc. Google, Facebook, Twitter, Yelp, Tripadvisor, Expedia, and Amazon all have business-facing websites providing guidance on participating in their sponsored advertising auctions.

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The motivation of this paper comes from that the previous literature on position auctions assumes each bidder only demands a single advertising slot. Moreover, most position auction formats, including the GSP auction and its variations,[3] as well as the generalized English auction,[4] are designed and studied based on this assumption. However, in some real-world position auctions, advertisers may have values over placing multiple sponsored advertisements in the same search result list. This is especially common in online retail platforms such as Amazon and eBay, where a lot of advertisers are multi-product suppliers selling differentiated products under the same category. Such suppliers may want to list several of their products in the same sponsored product list.[5] In addition to advertisers on online retail platforms, advertisers on online review and booking platforms may also demand multiple slots in the same sponsored business list.[6]

When bidders have values over getting multiple advertising slots, there is no reason to restrict each bidder from getting multiple slots under either surplus-maximizing or revenue-maximizing objective. While the current auction systems used by some platforms such as Google[7] and Amazon[8] already allow multiple advertisements from the same bidder to appear in the same sponsored search result list, the design of position auctions with multi-unit demand bidders is not well understood in the literature. How to design a position auction to allocate advertising slots efficiently when bidders have multi-unit demands is an interesting problem, as the auction needs to select both the quantity of slots allocated to each bidder and the rankings associated with each bidder's slots. This paper extends the study of position auctions by designing a two-stage ascending clock auction that allocates advertising slots efficiently among multi-unit demand bidders.

2. Related literature

This paper is related to a variety of strands in the literature. First, this paper is built upon the literature on position auctions. Among the earliest literature on position auctions, Edelman et al. (2007) and Varian (2007) study properties of GSP and VCG position auctions under complete information. Edelman et al. (2007) construct a dynamic position auction called the generalized English auction and show that this auction implements the VCG outcome in an ex post equilibrium. Since Edelman et al. (2007) and Varian (2007), many papers have studied position auctions under different modeling assumptions and design objectives, including Kominers (2009), Milgrom (2010), Edelman and Schwarz (2010), Athey and Ellison (2011), Chen and He (2011), Yenmez (2014), and Gomes and Sweeney (2014). All of the aforementioned papers assume that each advertiser only demands a single position. This paper contributes to this literature by extending position auctions into the more general multi-unit demand setting.

Second, this paper is related to the literature on efficient auction design. Early literature on efficient auction design includes the classic work of Vickrey (1961), Clarke (1971), and Groves (1973), who characterize a dominant strategy incentive compatible mechanism under pure private values. Ausubel et al. (2014) generalize the Vickrey auction into homogeneous item auctions with multi-unit demand bidders. Ausubel (2004)[9] constructs an efficient ascending-bid auction for homogeneous items that dynamically implements the VCG outcome in an ex post perfect equilibrium. In the Ausubel (2004) auction, a bidder "clinches" an item when the aggregate demand of its opponents falls below aggregate supply, and items are awarded at the price whenever they are "clinched." This paper generalizes the clinching rule in Ausubel (2004) into position auctions with multi-unit demand bidders. Ausubel (2006) proposes an efficient dynamic auction for selling multiple heterogeneous commodities. In the Ausubel (2006) auction, the auctioneer announces a price vector in each round, and each bidder responds by reporting one or more demand vectors that indicate its optimal consumption bundle(s) at the current price vector. The auctioneer then adjusts the price vector based on whether there is excess demand in a tatonnement process. Sincere bidding is an ex post perfect equilibrium of this auction. While the Ausubel (2006) auction can implement the efficient outcome under the context of position auctions, it requires using K clock prices to allocate K differentiated slots, and bidders are required to report K-dimensional demand vectors that indicate their demand for each of the K slots along the price path. Moreover, a parallel auction with multiple price paths[10] is required for implementing the VCG outcome in the Ausubel (2006) auction, which makes the auction procedure relatively complicated. This paper constructs an alternative

[3] The GSP auction has several variations, such as the GSP auction with quality-score adjusted bids and the GSP auction with pay-per-action payment rule (Edelman et al., 2007; Athey and Ellison, 2011).

[4] Edelman et al. (2007) introduce an ascending clock auction called the generalized English auction and show this auction dynamically implements the VCG outcome under pure private values.

[5] For example, a shoes supplier may want to list several of its products with different styles and prices in the sponsored product list for shoes on Amazon.

[6] For example, a hotel chain with multiple locations in the same geographical region may want to list several of its hotels on the same sponsored business list on Expedia.

[7] According to the Google Ads website, Google currently allows each advertiser to bid for multiple ads. When a user enters a search, Google's ads system finds all eligible ads that match the search and enters all corresponding bids into an auction. Therefore, when an advertiser places bids for multiple ads that are relevant to the same search, all of its ads that are identified to match the search are entered into auction when there's a relevant search.

[8] Similar to Google's ad system, Amazon's current ad auction system allows sellers to bid for multiple products and allows multiple bids from the same seller to be considered in the same sponsored product list.

[9] The dynamic auction for homogeneous items in Ausubel (2004) is referred to as "Ausubel auction" or "Ausubel (2004) auction" hereafter in this paper.

[10] In the Ausubel (2006) auction, if the initial price vector $\mathbf{p}(0)$ is chosen such that the market without bidder i clears at $\mathbf{p}(0)$, then bidder i receives its VCG payoff in an ex post perfect equilibrium. Without identical bidders, it is generally not possible to select an initial price vector $\mathbf{p}(0)$ such that every bidder receives its VCG payoff. Therefore, a parallel auction game with N price paths is required for computing the VCG payoff for every bidder when there are $N > 2$ non-identical bidders in the auction.

efficient dynamic auction that is tailored specifically to position auctions. The proposed auction can implement the VCG outcome using a single clock, which reduces the complexity of auction design under the special setting of position auctions.

Third, this paper connects with the literature on dynamic implementation of VCG outcome. In addition to Ausubel (2004, 2006), this strand of literature also includes work by Bikhchandani and Ostroy (2006), Mishra and Parkes (2007), Gul and Stacchetti (2000), Sun and Yang (2014), and Baranov (2018). de Vries et al. (2007) show that gross substitutes is a necessary condition for existence of ascending auctions to implement the VCG outcome. This paper contributes to this literature by designing a dynamic auction to implement the VCG outcome in allocating a set of vertically differentiated items, which is a special case of heterogeneous substitutable items.

3. Model

An auctioneer wishes to allocate K sponsored advertising positions among a set of N bidders. The N bidders are indexed by $i \in \{1, 2, \cdots, N\}$. The K advertising positions are indexed by $k \in \{1, 2, \cdots, K\}$. The click-through rate (CTR) profile of the advertising positions is given by $(\alpha_1, \alpha_2, \cdots, \alpha_K)$, in which $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_K$. The click-through rate profile is exogenously given and commonly known by all bidders.[11]

Each bidder i has value from receiving clicks on its advertisement(s) placed on the search result page. There is no restriction on the total number of advertisements that a single bidder can place. Each bidder can demand up to K units of advertising slots and can be assigned any quantity q_i of slots, in which $q_i \in \{0, 1, 2, \cdots, K\}$. Moreover, each bidder's value from receiving a click depends on whether the click is made on its first advertisement, its second advertisement, its third advertisement, etc. Bidders have diminishing marginal values over advertising slots, i.e., each bidder's value per click from its first advertisement is greater than its value per click from its second advertisement, which is in turn greater than its value per click from its third advertisement, etc.[12] Under this diminishing marginal value assumption, each bidder i's per-click value profile is given by a K-dimensional vector:

$$\mathbf{v}_i \equiv (v_i^1, v_i^2, \cdots, v_i^K), \quad v_i^1 \geq v_i^2 \geq \cdots \geq v_i^K \geq 0 \tag{1}$$

in which v_i^n denotes bidder i's value per click from its n-th advertisement in the search result list. Each bidder i knows its own values $\mathbf{v}_i$ but does not know other bidders' values $\mathbf{v}_{-i}$.

Assume bidders have quasilinear utilities. Let q_i denote the final quantity of positions assigned to bidder i. Let $\mathbf{r}_i \equiv (r_i^1, r_i^2, \cdots, r_i^{q_i}) \in \mathbb{R}^{q_i}$ denote the rankings associated with the q_i positions assigned to bidder i, in which r_i^1 is the ranking of the highest position assigned to bidder i, r_i^2 is the ranking of the second highest position assigned to bidder i, etc. Then bidder i's total payoff from winning q_i positions with ranking vector $\mathbf{r}_i$ at total payment p_i is given by

$$u(\mathbf{v}_i, q_i, \mathbf{r}_i, p_i) = \sum_{n=1}^{q_i} v_i^n \times \alpha_{r_i^n} - p_i \tag{2}$$

in which $\sum_{n=1}^{q_i} v_i^n \times \alpha_{r_i^n}$ is bidder i's total value from winning q_i positions with ranking vector $\mathbf{r}_i$. For example, if bidder i wins $q_i = 2$ positions with ranking vector $\mathbf{r}_i = (1, 4)$ at a total price of p_i, then its payoff from winning these two positions is given by $\alpha_1 v_i^1 + \alpha_4 v_i^2 - p_i$.

4. An efficient two-stage ascending clock auction

In this section, I characterize an efficient ascending clock auction that combines features from the Ausubel (2004) auction and the generalized English auction in Edelman et al. (2007). There are two stages in the proposed ascending clock auction: allocation stage and assignment stage. The allocation stage determines the quantity of advertising slots assigned to each bidder. The assignment stage determines the rankings associated with each winning bidder's slots.

[11] This paper assumes that, with multi-unit demand bidders, the click-through rate of an advertising slot depends only on the ranking of the slot and does not depend on how many advertisements the given advertiser has placed. This is a reasonable simplifying assumption when bidders demand multiple slots for their differentiated products and consumers have heterogeneous preferences over the same bidder's products. The click-through rate of a bidder's second advertisement is unlikely to be affected by its first advertisement or the total number of advertisements this bidder placed in the search list, as different advertisements would attract clicks from different groups of consumers. For many online platforms, clicks are largely driven by daily traffic and therefore can be viewed as exogenous.

[12] The diminishing marginal value assumption in this paper can be interpreted as follows. Each bidder's value from receiving a click on an ad can be interpreted as its expected profit from receiving a click on that specific ad. A bidder who places multiple ads can have different values per click among its own ads. For example, a TV supplier that sells multiple TV models can have a higher value per click from the advertisement on its newest model than its older models. An upscale restaurant with multiple locations can have a higher value per click from the advertisement on its location in high-income neighborhood than its other locations. Such bidder would always places its highest value per click advertisement in the first advertising slot it wins, places its second highest value per click advertisement in the second advertising slot it wins, etc. Therefore, it is reasonable to assume that a bidder's value per click from its first advertisement is greater than its value per click from its second advertisement, which is in turn greater than its value per click from its third advertisement, etc.

4.1. Allocation stage: an Ausubel auction with differentiated units

Consider an ascending clock auction with a single continuous clock price.[13] At any time t of the auction, let $p^t \in \mathbb{R}$ denote the clock price, and let $x_i^t \in \{0, 1, 2, \cdots, K\}$ denote the quantity of slots that each bidder i demands at p^t. Suppose the auction is fully subscribed at the initial price p^0, i.e., $\sum_i x_i^0 > K$, then the clock price increases continuously as long as the aggregate demand is strictly above the aggregate supply K, $\sum_i x_i^t > K$. The bidders can reduce their reported demand at any time of the auction. It is required that all bidders must bid monotonically:

$$x_i^{t'} \leq x_i^t \quad \text{for all } i, \text{for all } t' > t \tag{3}$$

The allocation stage ends at the earliest time $t = T$ when there is no excess demand, $\sum_i x_i^T \leq K$.

Following Ausubel (2004), a bidder i clinches $K - \sum_{j \neq i} x_j^t$ units of advertising slots at price p^t at the earliest time t when the aggregate demand of its opponents falls below the aggregate supply K. Bidder i clinches additional $\sum_{j \neq i} x_j^t - \sum_{j \neq i} x_j^{t'}$ units of slots at price $p^{t'}$ at the earliest time $t' > t$ when the aggregate demand of its opponents further falls below $\sum_{j \neq i} x_j^t$, etc. Formally, at any time t before the allocation stage ends, bidder i's cumulative clinches C_i^t is given by

$$C_i^t = \max\left\{0, K - \sum_{j \neq i} x_j^t\right\} \tag{4}$$

All bidders are constrained not to demand lower quantities than the quantities that they have already clinched:

$$x_i^{t'} \geq C_i^t \quad \text{for all } i, \text{for any } t' > t. \tag{5}$$

At the end of allocation stage, if $\sum_i x_i^T = K$, then there is no excess supply at p^T. In this scenario, each bidder i's final cumulative clinches C_i^T equals to its reported demand x_i^T, $C_i^T = x_i^T = K - \sum_{j \neq i} x_j^T$. If $\sum_i x_i^T < K$, then there is excess supply of $K - \sum_i x_i^T$ slots at p^T. In this scenario, the prioritized rationing rule proposed in Okamoto (2018) is used to allocate the excess supply. Each bidder i's final cumulative clinches C_i^T equals to its reported demand x_i^T at the end of allocation stage plus any additional units $\delta_i \geq 0$ allocated under the rationing rule, $C_i^T = x_i^T + \delta_i$. The design of the entire auction procedure, including its clinching rules and rationing rules, is publicly announced to all bidders at the beginning of the auction.

Under the prioritized rationing rule, each of the N bidders is assigned a randomly generated priority ranking. When there is excess supply at the end of allocation stage, the bidder with the highest priority ranking is assigned the extra units up to its lowest demand during the time $0 \leq t < T$. The bidder with the second highest priority ranking is assigned the remaining extra units up to its lowest demand during the time $0 \leq t < T$, etc. This process is repeated until all excess supply of $K - \sum_i x_i^T$ units are assigned. Formally, the bidder with priority ranking j is allocated C_j^T slots, in which

$$\begin{aligned}
&C_1^T = \min\{x_1^t, x_1^T + K - \sum_{i=1}^{N} x_i^T\}, \ \forall t < T \\
&C_2^T = \min\{x_2^t, x_2^T + K - \sum_{i=2}^{N} x_i^T - C_1^T\}, \ \forall t < T \\
&\cdots \\
&C_j^T = \min\{x_j^t, x_j^T + K - \sum_{i=j}^{N} x_i^T - \sum_{i=1}^{j-1} C_i^T\}, \ \forall t < T \\
&\cdots \\
&C_N^T = \min\{x_N^t, K - \sum_{i=1}^{N-1} C_i^T\}, \ \forall t < T
\end{aligned} \tag{6}$$

Under this rationing rule, the total number of slots assigned at the end of allocation stage always equals to K, i.e., $\sum_i C_i^T = K$. Once the allocation stage ends, the total quantity of slots assigned to each bidder i is final. No bidder can obtain any additional slots beyond its final cumulative clinches C_i^T in the assignment stage.

Since the advertising slots have different rankings and click-through rates, a few modifications on the allocation rule and payment rule are made when generalizing the Ausubel (2004) auction into the position auction setting. For each bidder i, when the aggregate demand of bidder i's opponents first falls below the aggregate supply K at clock price p^t, bidder

[13] All clock prices in this paper are per-click prices.

i clinches the **lowest ranked** C_i^t positions at a price of p^t per click. When the aggregate demand of bidder i's opponents further drops at a higher price $p^{t'} > p^t$, bidder i clinches the **next higher ranked** $C_i^{t'} - C_i^t$ positions above those C_i^t slots it has already clinched, at a price of $p^{t'}$ per click, etc. For example, a bidder clinches the lowest ranked position with α_K clicks at the price when the aggregate demand of its opponents falls from K to $K-1$ and clinches the second lowest ranked position with α_{K-1} clicks at the price when the aggregate demand of its opponents falls from $K-1$ to $K-2$. At the end of the allocation stage, each bidder i with $C_i^T > 0$ has cumulatively clinched the **lowest ranked** C_i^T positions.[14]

I next restrict attention to those bidders with $C_i^T > 0$ and formally define the payment rule in the allocation stage. For each bidder i with $C_i^T > 0$, suppose C_i^t jumps n times in the allocation stage. Let $t(1), t(2), \cdots, t(n)$ denote the time at which bidder i clinches a positive number of units. Then $0 < C_i^{t(1)} < C_i^{t(2)} < \cdots < C_i^{t(n)} = C_i^T$. At the end of the allocation stage, bidder i has clinched $C_i^{t(1)}$ lowest ranked positions at price $p^{t(1)}$, the next higher ranked $C_i^{t(2)} - C_i^{t(1)}$ positions at price $p^{t(2)}, \cdots$, the next higher ranked $C_i^{t(n)} - C_i^{t(n-1)}$ positions at price $p^{t(n)}$. The intuition of this modified clinching rule is that when the aggregate demand of opponents first falls below aggregate supply by $C_i^{t(1)}$ units, bidder i is guaranteed to win at least (and worst) the **lowest ranked** $C_i^{t(1)}$ positions at price $p^{t(1)}$, regardless of how its opponents bid in the rest of the auction. Similarly, when the aggregate demand of opponents further falls below aggregate supply by $C_i^{t(2)} - C_i^{t(1)}$ units at price $p^{t(2)}$, bidder i is guaranteed to win the **next higher ranked** $C_i^{t(2)} - C_i^{t(1)}$ positions at price $p^{t(2)}$, regardless of how its opponents bid in the rest of the auction, etc.

It follows that bidder i's total payment for its clinched positions in the allocation stage (Stage I) is given by

$$\begin{aligned} P_i^I &= \sum_{m=1}^{n} \sum_{k=K-C_i^{t(m)}+1}^{K-C_i^{t(m-1)}} \alpha_k p^{t(m)} \\ &= \sum_{k=K-C_i^{t(1)}+1}^{K} \alpha_k \times p^{t(1)} + \sum_{k=K-C_i^{t(2)}+1}^{K-C_i^{t(1)}} \alpha_k \times p^{t(2)} + \cdots + \sum_{k=K-C_i^{t(n)}+1}^{K-C_i^{t(n-1)}} \alpha_k \times p^{t(n)} \end{aligned} \tag{7}$$

I next present an example to illustrate how the allocation stage works.

4.2. An illustrative example of the allocation stage

Consider an auction with $K = 3$ positions and click-through rate profile $(\alpha_1, \alpha_2, \alpha_3) = (400, 300, 100)$. There are $N = 2$ advertisers, A and B. Both advertisers have values for getting more than one positions, and their marginal values are given in the following table. All numbers in the table are in dollar values.

	Bidder A	Bidder B
Marginal per-click value (first unit)	10	12
Marginal per-click value (second unit)	8	9
Marginal per-click value (third unit)	5	2

The allocation stage starts in the form of an ascending clock auction. Suppose the auction begins at price $p^0 = 0$. Both bidders A and B would respond with demand of 3 units at $p^0 = 0$. The aggregate demand of 6 units exceeds the aggregate supply of 3 units, so the price increases. Suppose that Bidder B reduces demand from 3 units to 2 units at the price of \$2, yielding:

Price	Bidder A	Bidder B	Aggregate demand	Allocation
2	3	2	5	A clinches Position 3

While the aggregate demand is still greater than the aggregate supply, from Bidder A's perspective, the total demand of its opponent is now 1 unit below the aggregate supply. Since all bidders are required to bid monotonically and no bidder will be allowed to obtain extra positions in the assignment stage, Bidder A is guaranteed to win at least 1 unit of

[14] When $\sum_i x_i^T < K$, each bidder i who receives a positive number of rationed units $\delta_i > 0$ "clinches" δ_i slots above the lowest ranked x_i^T positions at p^T per click. For example, if a bidder clinches one unit at $p^t = 2$ and is assigned an additional unit under the prioritized rationing rule at $p^T = 3$, then this bidder is considered to have clinched the lowest ranked position at the price of 2 per click and the second lowest ranked position at the price of 3 per click.

position in the final allocation. Even if Bidder A does not participate in the assignment stage, it is guaranteed to win the lowest ranked position with α_3 clicks. Therefore, Bidder A clinches Position 3 with $\alpha_3 = 100$ clicks at the price of \$2 per click.

Since there is still excess demand, price continues to increase. Suppose that Bidder A drops demand from 3 units to 2 units at the price of \$5, yielding:

Price	Bidder A	Bidder B	Aggregate demand	Allocation
5	2	2	4	B clinches Position 3

From Bidder B's perspective, the total demand of opponent is now 1 unit below the aggregate supply. Under the monotonic bidding rule, Bidder B is now guaranteed to win 1 unit of position no matter how Bidder A will bid later in the allocation stage. However, since Bidder A has the opportunity to upgrade its clinched units in the assignment stage, there is no guarantee for Bidder B to win any position ranked higher than Position 3 at this time. Hence, Bidder B clinches Position 3 with $\alpha_3 = 100$ clicks at the price of \$5 per click.

The clock price continues to increase given that there is still excess demand. Suppose that Bidder A further drops demand from 2 units to 1 unit at the price of \$8, yielding:

Price	Bidder A	Bidder B	Aggregate demand	Allocation
8	1	2	3	B clinches Position 2

Since the total demand of Bidder B's opponent is now 2 units below the total supply, Bidder B is now guaranteed to win 2 units of positions. Although Bidder A may upgrade its clinched position in the assignment stage, Bidder B is guaranteed to get at worst Position 2 for its first unit and at worst Position 3 for its second unit. Therefore, Bidder B clinches Position 2 with $\alpha_2 = 300$ clicks at the price of \$8 per click. Since the aggregate demand now equals the aggregate supply, the allocation stage ends at the price of \$8. The outcome of the allocation stage is summarized in the following table:

	Bidder A	Bidder B
Units clinched	1	2
Positions clinched	{Position 3}	{Position 2, Position 3}
Clicks clinched	100	300 + 100 = 400
Payment	$100 \times 2 = 200$	$100 \times 5 + 300 \times 8 = 2900$

4.3. Assignment stage: a generalized English auction with multi-unit demands

At the end of the allocation stage, each bidder i's final cumulative clinches C_i^T can be viewed as its "claims" for positions, and each bidder i's payment P_i^I in the allocation stage can be viewed as a payment for securing its claims. In the assignment stage, each bidder's payment in the allocation stage becomes a sunk cost. All active bidders take their claims into the assignment stage and compete for higher rankings for their clinched positions. The final ranking associated with each bidder's clinched positions is determined along with additional payments in the assignment stage. The design of the assignment stage is described below.

The assignment stage starts immediately from the ending price p^T of the allocation stage. At the beginning of the assignment stage, all bidders with positive demand $x_i^T > 0$ are considered to be active. If $\sum_i x_i^T < K$ at the end of allocation stage, then the lowest ranked $K - \sum_i x_i^T$ slots are assigned to those bidders who receive a positive number of slots under the prioritized rationing rule at time T.[15] All assignments in the assignment stage are final. If $\sum_i x_i^T = K$ at the end of allocation stage, no final assignment is made at time T.

At any time of the assignment stage, let p^t denote the current clock price in the auction. The clock price continues to increase when there are more than one active bidders in the auction. Each bidder may further reduce its demand as the price increases. Similar to the requirement in allocation stage, all bidders must bid monotonically in the assignment stage:

$$x_i^{t'} \le x_i^t \quad \text{for all } i, \text{for all } t' > t \tag{8}$$

[15] In the scenario when multiple slots are allocated under the rationing rule, i.e., $K - \sum_i x_i^T \ge 2$, and multiple bidders receive slots under the rationing rule, the exact ranking of these bidders within the lowest ranked $K - \sum_i x_i^T$ positions should be determined randomly by a lottery.

The first bidder who reduces demand by 1 unit gets the lowest ranked position above any rationed positions assigned at time T, as one of its clinched positions. The next bidder (who can be the same as the first bidder) who reduces demand by 1 unit gets the second lowest ranked position above any rationed positions assigned at time T, as one of its clinched positions, etc.[16] The auction ends when there is only one active bidder with strictly positive demand. That bidder gets the remaining unassigned position(s).

Let $r \in \{1, 2, \cdots, K\}$ denote the ranking of a position. Let $p(r)$ denote the clock price at which position r is assigned. Let q_i denote the total number of positions allocated to bidder i in the allocation stage. For every bidder i with $q_i > 0$, define the vector $(r_i^1, r_i^2, \cdots, r_i^{q_i})$ to be the ranking of positions assigned to bidder i in the assignment stage, with $r_i^1 < r_i^2 < \cdots < r_i^{q_i}$. Then r_i^1 is the highest ranking within the q_i slots assigned to bidder i; r_i^2 is the second highest ranking within the q_i slots assigned to bidder i, etc. For any $n \in \{1, 2, \cdots, q_i\}$, r_i^n is the ranking of bidder i's n-th highest position. According to the assignment rule, lower-ranked positions are assigned before higher-ranked positions, which implies $p(r_i^{q_i}) \le p(r_i^{q_i-1}) \le \cdots \le p(r_i^1)$.

Note that for every bidder i with $q_i > 0$, for any $n \in \{1, 2, \cdots, q_i\}$, the default ranking of bidder i's n-th highest position is $K - q_i + n$.[17] The price of the α_{K-q_i+n} clicks associated with the default ranking is already determined in the allocation stage. In order to upgrade the ranking of its n-th highest position from $K - q_i + n$ to $r_i^n \le K - q_i + n$, bidder i needs to pay an upgrading fee for those extra clicks beyond its clinched α_{K-q_i+n} clicks. The upgrading fee for bidder i's n-th highest unit is given by

$$\sum_{k=r_i^n}^{K-q_i+n-1} (\alpha_k - \alpha_{k+1}) p(k+1) \tag{9}$$

The intuition of this upgrading payment rule is that, each bidder i pays the total externalities imposed on its opponents by moving up each of its clinched positions, evaluated at corresponding losing bids.[18] For any k, the externalities that bidder i imposes on its opponents from moving up a position from $k+1$ to k is the value of additional clicks it receives from position k, evaluated at the price when position $k+1$ is assigned. The total externalities that bidder i imposes on its opponents from moving up its n-th position from $K - q_i + n$ to r_i^n is the sum of externalities associated with every step of move from $k+1$ to k, for $k = K - q_i + n - 1, K - q_i + n - 2, \cdots, r_i^n$, evaluated at relevant demand-reducing prices of opponents. Summing up the total externalities imposed on opponents from moving up all of its q_i positions, bidder i's total payment in the assignment stage (Stage II) is given by

$$P_i^{II} = \sum_{n=1}^{q_i} \sum_{k=r_i^n}^{K-q_i+n-1} (\alpha_k - \alpha_{k+1}) p(k+1) \tag{10}$$

Note that bidder i only pays an upgrading fee for the extra clicks resulted from moving up its slots, while its payment for clicks associated with the default ranking of each slot remains unchanged from the allocation stage. In this way, a bidder's payment for moving up a slot in the assignment stage is independent of its payment for claiming that slot in the allocation stage. This feature enables bidders to view how many slots to claim and whether to move up their slots as two independent decisions. I next present an example to illustrate how the assignment stage works.

4.4. An illustrative example of the assignment stage

Continue to consider the example in subsection 4.2. At the price $p^T = 8$, there is no excess demand in the auction. Bidder A has clinched 1 position with a default ranking of 3. Bidder B has clinched 2 positions with default rankings of 2 and 3, respectively. The auction will next proceed into assignment stage. In the assignment stage, the clock price will keep increasing from $p^T = 8$ until there is only 1 active bidder left in the auction.

When no bidder has reduced demand, both bidders are competing for upgrading their last unit from Position 3 to Position 2. From the perspective of each bidder, reducing demand by 1 unit means dropping out from the competition for

[16] If a bidder reduces demand by $n \ge 2$ units at the same price, it wins the current n lowest-ranked unassigned positions when no other bidder reduces demand at the same price. A tie-breaking lottery will be used to determine ranking when multiple bidders reduce demand at the same price. For example, if two bidders each reduce demand by 1 unit simultaneously and the rank of the lowest unassigned position at that time is given by r, then a tie-breaking lottery will be used to assign position r and $r-1$ to these two bidders. Each bidder should have 1/2 chance of receiving position r and 1/2 chance of receiving position $r-1$ under the tie-breaking lottery.

[17] To see why this is the case, first consider bidder i's q_i-th slot. The default ranking of this slot is always the lowest position K, according to the modified clinching rule in the allocation stage. Similarly, the default ranking of bidder i's $(q_i - 1)$-th slot is always the second lowest position $K-1$. The default ranking of bidder i's n-th slot is given by $K - q_i + n$.

[18] In an ex post equilibrium (characterized in section 5), all bidders reduce demand at their true marginal values. In equilibrium, the value of clicks evaluated at the demand-reducing price (i.e., highest losing bid for the higher ranked position) equal the value of clicks to the marginal losing bidder of the higher ranked position.

upgrading its last unit from Position 3 to Position 2. Suppose Bidder B reduces demand from 2 units to 1 unit at price of \$9, yielding:

Price	Bidder A	Bidder B	Number of active bidders	Assignment
9	1	1	2	B is assigned Position 3

As the first bidder who reduces demand, Bidder B will be assigned the lowest-ranked position, Position 3. Since Bidder B is guaranteed to win at least Position 3 as its last unit from the allocation stage, Bidder B does not need to pay any upgrading fee for winning Position 3.

Given that there are still 2 active bidders left in the auction, the clock price will continue to increase. After Position 3 is assigned, both bidders are now competing for upgrading their first unit from Position 2 to Position 1. From the perspective of each bidder, reducing demand by 1 means dropping out from the competition for upgrading its first unit from Position 2 to Position 1. Suppose Bidder A reduces demand from 1 unit to 0 unit at the price of \$10, yielding:

Price	Bidder A	Bidder B	Number of active bidders	Assignment
10	0	1	1	A is assigned Position 2 B is assigned Position 1

As the second bidder who reduces demand, Bidder A will be assigned the second-lowest ranked position, Position 2. Bidder A has upgraded its clinched position from Position 3 to Position 2. Bidder A's upgrading payment is given by

$$\begin{aligned} P_A^{II} =& (\alpha_2 - \alpha_3) \times p(3) \\ =& (300 - 100) \times 9 \\ =& 1800 \end{aligned} \tag{11}$$

Since there is only 1 active bidder left in the auction, the auction will end at price of \$10. Bidder B will be assigned the remaining position, Position 1, after Bidder A drops out. Bidder B has upgraded its first unit from Position 2 to Position 1. Bidder B's upgrading payment for winning Position 1 is given by

$$\begin{aligned} P_B^{II} =& (\alpha_1 - \alpha_2) \times p(2) \\ =& (400 - 300) \times 10 \\ =& 1000 \end{aligned} \tag{12}$$

The outcome of the assignment stage can be summarized in the following table:

	Bidder A	Bidder B
Positions clinched	{Position 3}	{Position 2, Position 3}
Positions assigned	{Position 2}	{Position 1, Position 3}
Extra clicks	$300 - 100 = 200$	$400 - 300 = 100$
Upgrading payment	$(300 - 100) \times 9 = 1800$	$(400 - 300) \times 10 = 1000$

The cumulative outcome of the auction is summarized in the following table:

	Bidder A	Bidder B
Units won	1	2
Positions won	{Position 2}	{Position 1, Position 3}
Clicks won	300	$400 + 100 = 500$
Total payment	$\underbrace{100 \times 2}_{\text{base price}} + \underbrace{(300 - 100) \times 9}_{\text{upgrading price}} = 2000$	$\underbrace{100 \times 5}_{\text{base price}} + \underbrace{300 \times 8}_{\text{base price}} + \underbrace{(400 - 300) \times 10}_{\text{upgrading price}} = 3900$

5. Equilibrium analysis

This section characterizes the ex post perfect equilibrium of the two-stage ascending clock auction.

First, consider a bidder's equilibrium strategy in the assignment stage given any outcome from the allocation stage. Let $\{q_i, x_i^T, \delta_i\}_{i=1}^N$ denote the outcome of allocation stage at time T, in which q_i is the total number of slots allocated to each bidder i, x_i^T is the reported demand of each bidder i at the end of allocation stage, and δ_i is the number of slots allocated to each bidder i under the rationing rule. By definition, $q_i = x_i^T + \delta_i$ for all i and $\sum_i q_i = K$.

At any time t of the assignment stage, the strategy of each active bidder i is specified by its reported demand $x_i(p^t, h^t)$ at the current clock price p^t, given history h^t. h^t contains the outcome of the allocation stage $\{q_i, x_i^T, \delta_i\}_{i=1}^N$ and the demand history up to time t. The following proposition characterizes each active bidder i's ex post perfect equilibrium strategy in the assignment stage.

Proposition 1. *In the assignment stage, at any price p^t, given any history h^t with outcome $\{q_i, x_i^T, \delta_i\}_{i=1}^N$ from the allocation stage, for any active bidder i with $x_i^T > 0$, define strategy $Q_i^{II}(p^t, h^t)$ as follows:*

$$Q_i^{II}(p^t, h^t) = \begin{cases} x_i^T & \text{if} \quad p^t \in (p^T, v_i^{x_i^T}) \\ x_i^T - 1 & \text{if} \quad p^t \in [v_i^{x_i^T}, v_i^{x_i^T - 1}) \\ x_i^T - 2 & \text{if} \quad p^t \in [v_i^{x_i^T - 1}, v_i^{x_i^T - 2}) \\ \cdots & \\ 1 & \text{if} \quad p^t \in [v_i^2, v_i^1) \\ 0 & \text{if} \quad p^t \in [v_i^1, \infty) \end{cases} \tag{13}$$

An ex post perfect equilibrium strategy for each bidder i with $x_i^T > 0$ at any time t of the assignment stage is given by

$$x_i^{II*}(p^t, h^t) = \min\left\{x_i^{t'}, Q_i^{II}(p^t, h^t)\right\}, \ \forall t' < t \tag{14}$$

The N-tuple strategy $x^{II*}(p^t, h^t) = \left(x_1^{II*}(p^t, h^t), x_2^{II*}(p^t, h^t), \cdots, x_N^{II*}(p^t, h^t)\right)$ is an ex post perfect equilibrium of the assignment stage.

Proof. See Appendix A. □

To better understand Proposition 1, consider the interpretation of demand reduction in the assignment stage. Since the quantity of positions assigned to each bidder is already determined in the allocation stage, there is no real "demand reduction" in the assignment stage. For each bidder i with $x_i^T > 0$, at any time in the assignment stage at which its demand remains at x_i^T, bidder i is deciding between getting the current lowest unassigned position as its x_i^T-th position and getting the next higher position as its x_i^T-th position with a higher upgrading fee. Reducing demand from x_i^T to $x_i^T - 1$ implies that bidder i prefers getting the current lowest unassigned position as its x_i^T-th position over further upgrading its x_i^T-th position to the next higher rank. Similarly, for any $n \in \{1, 2, \cdots, x_i^T\}$, reducing demand from n to $n-1$ implies that bidder i prefers getting the current lowest unassigned position as its n-th highest position over further upgrading its n-th highest position to the next higher rank. This is the correct interpretation of demand reduction in the assignment stage.

Proposition 1 comes from the fact that, given any outcome in the allocation stage, with any demand-reducing history h^t, each active bidder i with $x_i^T > 0$ would be willing to upgrade its x_i^T-th position until the clock price reaches $v_i^{x_i^T}$, the value from receiving an extra click on its x_i^T-th advertisement. When the clock price exceeds $v_i^{x_i^T}$, the payoff from upgrading its x_i^T-th position from the current lowest unassigned position to the next higher-ranked position becomes negative. Therefore, assuming all opposing bidders are following the same strategy, it is optimal for bidder i to reduce its reported demand from x_i^T to $x_i^T - 1$ at clock price $v_i^{x_i^T}$. Similarly, it is optimal for bidder i keep its reported demand at $x_i^T - 1$ when the clock price is in the interval $[v_i^{x_i^T}, v_i^{x_i^T-1})$, during which the payoff from upgrading its $(x_i^T - 1)$-th position from the current lowest unassigned position to the next higher position is strictly positive. Under the same reasoning, for every $n \in \{1, 2, \cdots, x_i^T\}$, assuming all opposing bidders are following the equilibrium strategy, bidder i can maximize its ex post payoff by keeping demand at n when the clock price falls in the interval $[v_i^{n+1}, v_i^n)$, subject to the monotonic bidding requirement. If a bidder makes a mistake by reducing demand too early in the auction, its optimal strategy after the mistake would be to keep its demand until the clock price rises to its marginal value for the next slot.[19] The minimum function in equation (14) ensures

[19] For example, suppose a bidder reduces demand from 2 to 1 at $p^t = 10$, while its marginal value profile is $(v_i^1, v_i^2) = (20, 15)$. An ex post equilibrium strategy for this bidder is to keep its demand at 1 when $10 \leq p^t < 20$ and drop out at $p^t = 20$.

that the equilibrium strategy at any time t is well specified under any mistakes that bidder i might have made before time t.

Next, consider each bidder's equilibrium strategy in the allocation stage. At any time t in the allocation stage, each bidder's strategy specifies its reported demand at current clock price p^t, given any demand-reducing history h^t up to time t in the allocation stage. The following proposition characterizes each bidder i's ex post perfect equilibrium strategy in the allocation stage.

Proposition 2. *In the allocation stage, at any price p^t, given any history h^t, for each bidder i, define strategy $Q_i^I(p^t, h^t)$ as follows:*

$$Q_i^I(p^t, h^t) = \begin{cases} K & \text{if} \quad p^t \in [p^0, v_i^K) \\ K-1 & \text{if} \quad p^t \in [v_i^K, v_i^{K-1}) \\ K-2 & \text{if} \quad p^t \in [v_i^{K-1}, v_i^{K-2}) \\ \cdots & \\ 1 & \text{if} \quad p^t \in [v_i^2, v_i^1) \\ 0 & \text{if} \quad p^t \in [v_i^1, \infty) \end{cases} \tag{15}$$

An ex post perfect equilibrium strategy for each bidder i at any time t in the allocation stage is given by

$$x_i^{I*}(p^t, h^t) = \min\left\{x_i^{t'}, \max\left\{Q_i^I(p^t, h^t), C_i^{t'}\right\}\right\}, \ \forall t' < t \tag{16}$$

The N-tuple strategy $x^{I*}(p^t, h^t) = \left(x_1^{I*}(p^t, h^t), x_2^{I*}(p^t, h^t), \cdots, x_N^{I*}(p^t, h^t)\right)$ is an ex post perfect equilibrium of the allocation stage.

Proof. See Appendix A. □

The intuition of Proposition 2 comes from that each bidder only needs to evaluate the trade-off between winning different quantities of slots in the allocation stage, leaving the upgrading decision to the assignment stage. At any time t of the allocation stage, if a bidder i reduces its reported demand from n to $n-1$, then it must be indifferent between winning n slots and winning $n-1$ slots at p^t. It follows that for any $n \in \{1, 2, \cdots, K\}$, assuming all opposing bidders follow the same strategy, it is optimal for bidder i to keep demand at n until price reaches v_i^n, at which it is indifferent between winning n slots and winning $n-1$ slots. Since bidders have the chance to upgrade their rankings in the assignment stage, the ranking of positions does not play a role in the allocation stage. Each bidder views all positions as homogeneous items when choosing its strategy in the allocation stage. Therefore, bidders' incentives in the allocation stage are similar to the bidders' incentives in the Ausubel (2004) auction, in which sincere bidding by reporting true demand schedules is an ex post perfect equilibrium.

Having shown that $x^{II*}(p^t, h^t)$ is an ex post perfect equilibrium of the assignment stage given *any* outcome $\{q_i, x_i^T, \delta_i\}_{i=1}^N$ from the allocation stage, and that $x^{I*}(p^t, h^t)$ is an ex post perfect equilibrium of the allocation stage, I next show that given the equilibrium strategy of assignment stage, $x_i^{I*}(p^t, h^t)$ also maximizes each bidder i's ex post payoff in the entire auction by selecting the optimal x_i^{T*} that maximizes its ex post equilibrium payoff in the assignment stage, which yields the following proposition:

Proposition 3. *At any time t, given any history h^t, each bidder i's ex post perfect equilibrium strategy in the two-stage ascending clock auction is given by*

$$x_i^*(p^t, h^t) = \begin{cases} x_i^{I*}(p^t, h^t) & \text{when} \quad p^t \in [p^0, p^T] \\ x_i^{II*}(p^t, h^t) & \text{when} \quad p^t \in [p^T, \infty] \end{cases} \tag{17}$$

in which p^T is the market clearing price that satisfies $\sum_i x_i^{I*}(p^t, h^t) > K$ for all $t < T$ and $\sum_i x_i^{I*}(p^T, h^T) \leq K$. The N-tuple strategy $x^*(p^t, h^t) = \left(x_1^*(p^t, h^t), x_2^*(p^t, h^t), \cdots, x_N^*(p^t, h^t)\right)$ is an ex post perfect equilibrium of the two-stage ascending clock auction.

Proof. See Appendix A. □

Proposition 3 demonstrates that sincere bidding by all bidders through reporting true marginal value schedules is an ex post perfect equilibrium in the two-stage ascending clock auction. It follows that the equilibrium outcome is always efficient: the two-stage ascending clock auction always assigns more positions and higher ranked positions to bidders with higher marginal values. Given the sincere bidding equilibrium and the payment rules (7) and (10), it is straightforward to see that the two-stage ascending clock auction yields the VCG outcome under pure private values.

Corollary 1. *The two-stage ascending clock auction dynamically implements the VCG outcome in an ex post perfect equilibrium under pure private values.*

The intuition behind the outcome equivalence of the two-stage ascending clock auction and the VCG mechanism is the follows. In the VCG mechanism, each winning bidder pays the total externalities it imposes on all opponents. Under the position auction setting, this total externalities can be decomposed into two components. First, each winning bidder imposes externalities on its opponents by depriving a number of slots from its opponents. If a winning bidder did not participate in the auction, some of its opponents would have won more slots. Second, each winning bidder imposes externalities on its opponents by depriving higher ranked positions from its opponents. If a winning bidder did not participate in the auction, some of its opponents would have won higher ranked positions. In the two-stage ascending clock auction, the payment rule in the allocation stage captures the first type of externalities, while the payment rule in the assignment stage captures the second type of externalities. In the allocation stage, the payment rule (7) ensures that each bidder pays the value of clicks associated with the positions it deprives from opponents before upgrading, evaluated at the highest losing bid for each unit of position. In the assignment stage, the payment rule (10) ensures that each bidder pays the value of extra clicks associated with the higher rankings it deprives from opponents, evaluated at the highest losing bid for each ranking. Therefore, the two-stage ascending clock auction replicates the VCG outcome in equilibrium.

Note that in the special case when each bidder only demands a single unit of position, i.e., $\mathbf{v}_i = (v_i^1, 0, 0, \cdots, 0)$ for all i, the two-stage ascending clock auction breaks down to the generalized English auction, in which each bidder drops out at its true value v_i^1 in an ex post perfect equilibrium, consistent with the result in Edelman et al. (2007). In the special case when all positions have identical click-through rates, i.e., $\alpha_1 = \alpha_2 = \cdots = \alpha_K$, the two-stage ascending clock auction breaks down to the Ausubel (2004) auction, in which each bidder reports its true demand schedule in an ex post perfect equilibrium, consistent with the result in Ausubel (2004). Therefore, the two-stage ascending clock auction nests the generalized English auction and the Ausubel (2004) auction as two special cases. It generalizes the outcome equivalence between the generalized English auction and the VCG position auction into the multi-unit demand setting and generalizes the outcome equivalence between the Ausubel (2004) auction and the multi-unit Vickrey auction into the position auction setting.

The two-stage ascending clock auction has desirable incentive properties in the following senses. First, each bidder's equilibrium strategy does not depend on the distribution of opposing bidders' values. Bidders do not need to worry about ex post regret under any realization of opposing bidders' values. Second, each bidder's equilibrium strategy is robust under any click-through rate profile, including an extended setting where the click-through rate of each position depends on the number of advertisements that the same bidder has placed above that position.

Corollary 2. *Consider an extended setting where the click-through rate of each position depends on the number of positions that the same bidder has above that position. Under this extended setting, sincere bidding by all bidders is still an ex post perfect equilibrium of the two-stage ascending clock auction, but this equilibrium can be inefficient.*

The robustness of sincere bidding equilibrium under this extended setting comes from the fact that all payments are per-click payments in the two-stage ascending clock auction. When a bidder decides whether to reduce demand in the auction, it only needs to consider whether its marginal value *per click* exceeds the price *per click* of the extra clicks associated with an additional slot or a higher ranking, regardless of whether the number of extra clicks depends on the number of positions it already wins. The number of clicks associated with each position does not enter each bidder's strategic consideration problem. Sincere bidding is an ex post equilibrium strategy under any click-through rate profile.

Note that the two-stage ascending clock auction loses its efficiency properties when the click-through rate of each position depends on its rank relative to the same bidder's other advertisements. To see why this is the case, consider an example with 2 positions and 2 bidders, i and j. The click-through rate of Position 1 is 200. The click-through rate of Position 2 is 150 if it's the first position assigned to a bidder, while the click-through rate of Position 2 is 100 if it's the second position assigned to a bidder. Suppose $(v_i^1, v_i^2) = (2, 1.1)$ and $(v_j^1, v_j^2) = (1, 0)$. Bidder j's equilibrium strategy is to reduce demand to 0 at $p^t = 1$, since it is unwilling to pay more than \$1 per click for any position. Similarly, bidder i's equilibrium strategy is to keep demand at 2 units until $p^t = 1.1$, since it has a strictly positive payoff per click from both positions until price reaches \$1.1. Both positions are assigned to bidder i in an ex post equilibrium, yielding a total surplus of $2 \times 200 + 1.1 \times 100 = 510$. In an alternative outcome where Position 1 is assigned to bidder i and Position 2 is assigned to bidder j, the total surplus is $2 \times 200 + 1 \times 150 = 550$. Therefore, the sincere bidding equilibrium is no longer efficient under this extended setting. This loss of efficiency comes from the fact that under the per-click payment rule, bidders are unable to incorporate diminishing marginal number of clicks from winning additional positions into their bidding strategies. How to design an efficient auction when the click-through rate of each advertisement depends on the number of advertisements the same bidder has already placed is an open question for future research.

While sincere bidding is an ex post equilibrium strategy in the two-stage ascending clock auction, it is not a dominant strategy or an obviously dominant strategy defined in Li (2017).[20] Consider the following example with 3 positions and 2 bidders, i and j. Bidder i's marginal value profile is given by $(v_i^1, v_i^2, v_i^3) = (5, 1, 0)$. The click-through rate profile is given by $CTR = (101, 1, 1)$. Suppose that at time t in the assignment stage, $p^t = 0.5$, bidder i demands 2 slots and bidder j demands 1 slot. Suppose bidder j's strategy is to keep demand at 1 when bidder i demand 2 slots and to reduce demand

[20] Obvious dominance implies dominance, so showing that sincere bidding is not a dominant strategy is sufficient for proving it's not an obviously dominant strategy.

to 0 immediately after bidder i reduces demand by any amount. If bidder i plays a deviating strategy by reducing demand from 2 to 1 immediately at $p^t = 0.5$, it will win position 1 and 3 and pay $\$0.5 \times 100 = \$50$ upgrading fee. If bidder i bids sincerely by keeping demand at 2 until clock price reaches its true marginal value, $p^{t'} = 1$, where $t' > t$, it will still win position 1 and 3, but pay a higher upgrading fee of $\$1 \times 100 = \$100$. Therefore, there exists opposing bidder's strategy under which sincere bidding does not maximize bidder i's ex post payoff. Ex post perfect equilibrium is the appropriate equilibrium concept for the two-stage ascending clock auction.[21]

6. Conclusions

Previous literature on position auctions has restricted attention to single-unit demand models. However, bidders may have value over placing multiple advertisements in the same sponsored advertisement list. This is especially true for multi-product suppliers selling differentiated products in online marketplaces and for business chains providing services at different locations in online booking platforms. This paper constructs an efficient dynamic position auction in a multi-unit demand setting. The proposed auction implements the VCG outcome in an ex post perfect equilibrium under pure private values.

One implication of this paper is that the VCG auction can be generalized into the multi-unit demand setting, while the commonly-used GSP auction has no direct generalization with multi-unit demand bidders. The next highest bid payment rule in the standard GSP auction is likely to cause efficiency loss under the multi-unit demand setting, as bidders can have strategic incentives to avoid paying their own bids on lower-ranked positions for higher-ranked positions. Therefore, this paper adds to the literature that favors the VCG auction over the GSP auction for its better adaptability under environments outside Edelman et al. (2007), such as incomplete information and unknown click-through rates (Gomes and Sweeney, 2014; Varian and Harris, 2014).

Another implication of this paper is that, under the special structure of position auctions, a single clock is sufficient for allocating multiple differentiated positions efficiently to multi-unit demand bidders. This result echoes the literature that recognizes the simplicity of GSP and VCG position auctions in the single-unit demand setting[22] (Milgrom, 2010) and demonstrates that the existence of efficient position auctions with single-dimensional bids does not rely on the single-unit demand assumption. The fact that only a single clock is sufficient for allocating heterogeneous positions is driven by a special feature of position auctions: the heterogeneity across positions can be represented in a single-dimensional attribute, click-through rates. That is, click-through rate is the only attribute that bidders care about for any position in the auction. A bidder's value for any position can be calculated by scaling its marginal value per click from an additional advertisement by the click-through rate of the position. This feature reduces the dimensionality of the allocation problem and enables the auctioneer to use a single clock to assign multiple positions. The single-dimensional clock price reflects the price of each click, not the price of any specific position. It follows that a single clock is sufficient for allocating heterogeneous positions efficiently.

There are a few potential future research directions relevant to this paper. First, prior literature has pointed out the analogy between position auctions and two-sided matching markets while restricting attention to one-to-one matching markets (Edelman et al., 2007; Johnson, 2013). In practice, two-sided many-to-one matching has become increasingly common in many two-sided platforms, such as carpool matching on mobile ride apps. A natural extension of this paper is to explore the possibility of using position auction-like mechanisms to improve match surplus and revenue in two-sided many-to-one matching markets.

Second, prior literature has studied position auctions with budget constraints (Ashlagi et al., 2010) while restricting bidders to have single-unit demand. Another potential extension of this paper is to introduce budget constraints to the model and allow bidders to allocate budget over multiple slots.

Appendix A

Proof of Proposition 1. At the beginning of assignment stage, the lowest ranked $K - \sum_{j=1}^{N} x_j^T \geq 0$ slots are assigned to those bidders who receive slots under the rationing rule. Only the top ranked $\sum_{j=1}^{N} x_j^T$ slots are considered in the assignment problem. For each bidder i with $x_i^T > 0$, the default rankings associated with these x_i^T units is given by $\sum_{j=1}^{N} x_j^T, \sum_{j=1}^{N} x_j^T - 1, \cdots, \sum_{j=1}^{N} x_j^T - x_i^T + 1$. That is, bidder i is guaranteed to win at worst the $\sum_{j=1}^{N} x_j^T$-th position for its lowest ranked slot among its top x_i^T slots, at worst the $(\sum_{j=1}^{N} x_j^T - 1)$-th position for its second lowest ranked slot among its top x_i^T slots, etc. For any $n \in \{1, 2, \cdots, x_i^T\}$, the default ranking of bidder i's n-th highest ranked slot among its top x_i^T slots is given by $\sum_{j=1}^{N} x_j^T - x_i^T + n$.

[21] This is not unique to the setting of position auctions with multi-unit demands. In both Ausubel (2004) auction and generalized English auction, sincere bidding is an ex post equilibrium strategy but not a dominant strategy or an obviously dominant strategy.

[22] In standard GSP and VCG position auctions with single-unit demand bidders, only single-dimensional bids are required to allocate K differentiated positions, while K-dimensional bids are required to allocate K heterogeneous items in a standard VCG mechanism under more general settings.

Consider the strategy of any active bidder i at any time t in the assignment stage. Suppose bidder i's current reported demand is given by n and the ranking of the lowest unassigned position is given by r. At time t, bidder i needs to decide whether to reduce demand now and get the r-th position as its n-th highest ranked slot among its top x_i^T slots, or to keep demand at n in order to further upgrade its n-th highest ranked slot to $r-1$. Note that bidder i will be able to win position $r-1$ at an upgrading fee of b per click if another bidder reduces demand at price b. In equilibrium, assuming all opponents are following the same strategy, bidder i should be indifferent between winning position $r-1$ as its n-th position with an upgrading fee b per click for the extra $\alpha_{r-1}-\alpha_r$ clicks and winning position r as its n-th position, which gives the following equilibrium condition:

$$\alpha_{r-1} v_i^n - (\alpha_{r-1}-\alpha_r) b - P_{i,n,r}^{II} - P_{i,n}^{I} = \alpha_r v_i^n - P_{i,n,r}^{II} - P_{i,n}^{I} \tag{18}$$

in which $P_{i,n,r}^{II}$ is the upgrading fee of moving bidder i's n-th highest position from its default ranking of $\sum_{j=1}^N x_j^T - x_i^T + n$ at time T to its new default ranking of r at time t. $P_{i,n}^I$ is bidder i's payment for winning its n-th highest slot in the allocation stage. Note that bidder i has already upgraded the ranking of its n-th position to rank r at time t when the current lowest unassigned position is r. Therefore, the existing upgrading fee $P_{i,n,r}^{II}$ and the allocation stage payment $P_{i,n}^I$ are sunk costs at time t. Bidder i only needs to decide whether to further upgrade its n-th position given the current clock price. For any $n \in \{1, 2, \cdots, x_i^T\}$, for any $r \in \{1, 2, \cdots, \sum_{j=1}^N x_j^T\}$, given any history h^t, the equilibrium condition can always be simplified to

$$\alpha_{r-1} v_i^n - (\alpha_{r-1}-\alpha_r) b = \alpha_r v_i^n \tag{19}$$

which gives $b = v_i^n$.

Therefore, assuming all opponents are following the same strategy, at any time t, given any history h^t, bidder i can always maximize its ex post payoff at the assignment stage by keeping its reported demand at n until the clock price reaches v_i^n, for all $n \in \{1, 2, \cdots, x_i^T\}$. Define strategy $Q_i^{II}(p^t, h^t)$ as follows:

$$Q_i^{II}(p^t, h^t) = \begin{cases} x_i^T & \text{if} \quad p^t \in [p^T, v_i^{x_i^T}) \\ x_i^T - 1 & \text{if} \quad p^t \in [v_i^{x_i^T}, v_i^{x_i^T-1}) \\ x_i^T - 2 & \text{if} \quad p^t \in [v_i^{x_i^T-1}, v_i^{x_i^T-2}) \\ \cdots \\ 1 & \text{if} \quad p^t \in [v_i^2, v_i^1) \\ 0 & \text{if} \quad p^t \in [v_i^1, \infty) \end{cases} \tag{20}$$

At any time t in the assignment stage, bidder i's ex post perfect equilibrium strategy $x_i^{II*}(p^t, h^t)$ is to follow $Q_i^{II}(p^t, h^t)$, subject to the constraints posed by the monotonic activity rule. Hence, each bidder i's ex post perfect equilibrium bidding strategy in the assignment stage is given by

$$x_i^{II*}(p^t, h^t) = \min\left\{x_i^{t'}, Q_i^{II}(p^t, h^t)\right\}, \ \forall t' < t \quad \square \tag{21}$$

Proof of Proposition 2. For any bidder i, at any time t such that i has a strictly positive demand at n, given any history up to t, bidder i's problem is to decide whether to reduce demand below n at the current clock price p^t. Suppose bidder i keeps demand to be $x_i(p^t, h^t) = n$, then there is a positive probability that $\sum_{j\neq i} x_j(p^t, h^t) < \sum_{j\neq i} x_j(p^{t-\epsilon}, h^{t-\epsilon})$ for some arbitrarily small ϵ and $\sum_{j\neq i} x_j(p^t, h^t) < K$ occurs at time t, making bidder i clinch some units at price p^t. Bidder i needs to ensure that its ex post payoff per click from winning an n-th unit at the current clock price p^t is positive, i.e., $v_i^n - p^t \geq 0$. Therefore, at any time of the auction, for any $n \in \{1, 2, \cdots, K\}$, it is optimal for bidder i to keep demand at n until price reaches v_i^n, the value from receiving a click on its n-th advertisement. Assuming all opponents follow the same strategy, then this strategy maximizes bidder i's ex post payoff at the end of the allocation stage given any history h^t. Define strategy $Q_i^I(p^t, h^t)$ as follows:

$$Q_i^{I}(p^t, h^t) = \begin{cases} K & \text{if} \quad p^t \in [p^0, v_i^K) \\ K - 1 & \text{if} \quad p^t \in [v_i^K, v_i^{K-1}) \\ K - 2 & \text{if} \quad p^t \in [v_i^{K-1}, v_i^{K-2}) \\ \cdots \\ 1 & \text{if} \quad p^t \in [v_i^2, v_i^1) \\ 0 & \text{if} \quad p^t \in [v_i^1, \infty) \end{cases} \tag{22}$$

At any time t in the allocation stage, each bidder i's ex post perfect equilibrium strategy is to follow $Q_i^I(p^t, h^t)$, subject to the constraints that all bidders must bid monotonically, and bidders cannot reduce demand below their already clinched units. Hence, each bidder i's ex post perfect equilibrium strategy in the allocation stage is given by

$$x_i^{I*}(p^t, h^t) = \min\left\{x_i^{t'}, \max\{Q_i^I(p^t, h^t), C_i^{t'}\}\right\}, \ \forall t' < t \quad \square \tag{23}$$

Proof of Proposition 3. The proof of Proposition 1 shows that for each bidder i, given *any* outcome $\{q_i, x_i^T, \delta_i\}_{i=1}^N$ from the allocation stage, assuming all opponents follow the same equilibrium strategy, $x_i^{II*}(p^t, h^t)$ always maximizes bidder i's ex post payoff at the end of assignment stage. Moreover, given any outcome $\{q_i, x_i^T, \delta_i\}_{i=1}^N$, each bidder's equilibrium strategy $x_i^{II*}(p^t, h^t)$ in the assignment stage depends only on its own final demand x_i^T and does not depend on any other information contained in $\{q_i, x_i^T, \delta_i\}_{i=1}^N$. It only leaves to prove that given correct expectation about the assignment stage, each bidder i's equilibrium strategy $x_i^{I*}(p^t, h^t)$ in the allocation stage yields an optimal final demand x_i^{T*} that maximizes bidder i's final ex post payoff at the end of the two-stage auction. Assuming all of bidder i's opponents follow strategy $x_j^{I*}(p^t, h^t)$ in the allocation stage and follow strategy $x_j^{II*}(p^t, h^t)$ in the assignment stage, I next show that deviating from $x_i^{I*}(p^t, h^t)$ in the allocation stage can only hurt bidder i's final ex post payoff at the end of the assignment stage.

For any $n \in \{1, 2, \cdots, K\}$, if bidder i reports $x_i(p^t, h^t) < n$ when $p^t < v_i^n$ in the allocation stage, then there is a positive probability that bidder i gets fewer than n slots at the end of the allocation stage while it could have clinched an n-th slot at price $p^t < v_i^n$, earning a positive payoff per click from this slot. There is nothing that bidder i can do in the assignment stage to compensate for this loss, as it is impossible to win additional units of advertising slots in the assignment stage. On the other hand, if bidder i reports $x_i(p^t, h^t) \geq n$ when $p^t > v_i^n$, there is a positive probability that bidder i clinches an n-th slot at $p^t > v_i^n$, earning a negative payoff per click. There is nothing that bidder i can do in the assignment stage to compensate for this loss, as the best thing that bidder i can do is to not to upgrade this slot from its default ranking. Therefore, the equilibrium strategy $x_i^{I*}(p^t, h^t)$ always yields the optimal x_i^{T*} for bidder i at the end of allocation stage.

Since each bidder's equilibrium strategy $x_i^{II*}(p^t, h^t)$ maximizes its ex post payoff given any final demand x_i^T from the allocation stage, deviating from $x_i^{I*}(p^t, h^t)$ in the allocation stage and getting a quantity different from x_i^{T*} can only reduce bidder i's final ex post payoff. $x_i^{I*}(p^t, h^t)$ not only maximizes bidder i's ex post payoff of the allocation stage but also maximizes bidder i's ex post payoff of the entire auction by selecting the optimal final demand x_i^{T*} for the assignment stage. Hence, each bidder i's ex post perfect equilibrium bidding strategy in the two-stage ascending clock auction can be expressed as

$$x_i^*(p^t, h^t) = \begin{cases} x_i^{I*}(p^t, h^t) & \text{when} \quad p^t \in [p^0, p^T] \\ x_i^{II*}(p^t, h^t) & \text{when} \quad p^t \in [p^T, \infty] \end{cases} \tag{24}$$

where p^T is the market clearing price that marks the end of the allocation stage, i.e., $\sum_i x_i(p^t, h^t) > K$ for all $t < T$, and $\sum_i x_i(p^t, h^t) \leq K$ for all $t \geq T$. $\square$

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