Position Auctions with Interdependent Values

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Outline

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Model

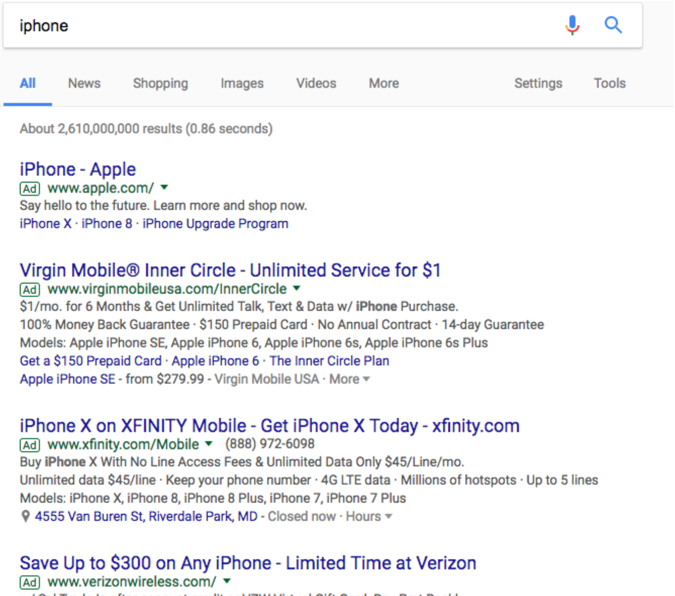
Main Results: Efficiency

Main Results: Revenue

Conclusions

Introduction

An Example of Sponsored Search Advertising



Standard Framework of Position Auctions

Standard Framework (Edelman et al. 2007; Varian 2007)

- K advertising positions; $N > K$ bidders.
- Positions differ in click-through-rate (CTR): $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_K$ are exogenous and commonly known.
- Advertisers differ in value per click, v_i.
- Advertiser i's total value of the k-th highest position is $\alpha_k \times v_i$.

Three Position Auction Formats

- Generalized Second Price Auctions (GSP): $p_{(k)} = \alpha_k b_{(k+1)}$
- Vickrey-Clarke-Groves Auctions (VCG): $p_{(k)} = \sum_{j=k}^{K} (\alpha_j - \alpha_{j+1}) b_{(j+1)}$
- Generalized English Auctions (GEA): ascending clock auction, $p_{(k)} = \alpha_k b_{(k+1)}$

Motivation: Interdependent Values

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- There exists a common component in all advertisers' values $(v_1, v_2, \cdots, v_N)$ that is driven by aggregate demand.
- Suppose each advertiser receives a private signal x_i that estimates how likely consumers are going to purchase its product after click.
- Both x_i and other advertisers' signals x_{-i} are informative about v_i.

Contribution

Research Questions

In an interdependent values model:

- Are GSP, VCG and GEA still efficient? If not, how to improve efficiency?
- How do the revenues of GSP, VCG and GEA compare?
- What is the optimal (revenue-maximizing) auction? How do the revenues of GSP, VCG and GEA compare to the optimal revenue?

Main Contribution

- Extend the study of three standard position auctions into interdependent values.
- Propose two new auction mechanisms to improve efficiency and revenue.

Summary of Results: Efficiency

Previous Literature - Under Complete Information:

- GSP, VCG and GEA are all efficient.

This Paper - Under Interdependent Values:

- Both GSP and VCG can be inefficient. GEA is always efficient.

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- Both GSP and VCG can be inefficient. GEA is always efficient.
- I propose a modification of GSP and VCG by allowing bidders to condition their bids on positions.
- Both K-dimensional GSP and K-dimensional VCG are efficient.

Summary of Results: Revenue

Previous Literature - Under Complete Information:

- Revenue ranking: GSP $\geq$ VCG $=$ GEA

This Paper - Under Interdependent Values:

- Revenue ranking: GEA $\geq$ K-dimensional VCG $=$ K-dimensional GSP

Summary of Results: Revenue

Previous Literature - Under Complete Information:

- Revenue ranking: GSP $\geq$ VCG $=$ GEA

This Paper - Under Interdependent Values:

- Revenue ranking: GEA $\geq$ K-dimensional VCG $=$ K-dimensional GSP
- Under independent signals, the GEA, K-dimensional GSP and K-dimensional VCG are revenue equivalent and implement the optimal revenue subject to no reserve price.

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- The signals $x = (x_1, x_2, \cdots, x_N)$ are distributed according to joint distribution $F(x_1, x_2, \cdots, x_N)$ with density $f(x_1, x_2, \cdots, x_N)$.

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- Bidder i's value per click is $v_i(x_i, x_{-i})$. $v_i(.,.)$ symmetric across bidders.
- Quasilinear utility:

$$U_i(x_i, x_{-i}, k) = \alpha_k v_i(x_i, x_{-i}) - p^{(k)}$$

Assumptions

- **A1** $v(x_i, x_{-i})$ is nonnegative, continuous and strictly increasing in x_i, nondecreasing in x_j.

$$\frac{\partial v_i(x_i, x_{-i})}{\partial x_i} > 0, \frac{\partial v_i(x_i, x_{-i})}{\partial x_j} \geq 0, \quad \forall j \neq i$$

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- **A4** $f(x_1, x_2, \cdots, x_N)$ is symmetric in all arguments.
- **A5** The signals $x_1, x_2, \cdots, x_N$ are affiliated: For any x and x':

$$f(x \vee x')f(x \wedge x') \geq f(x)f(x')$$

The Generalized Winner's Curse and Efficiency

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A position auction is efficient if it always assigns positions in the rank ordering of bidders' ex-post values.

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- $v^k(x_i, y_k)$: expected value per click conditional on realizations of X and Y_k:
$$v^k(x_i, y_k) = E\left[v(x_i, x_{-i}) \middle| X = x_i, Y_k = y_k\right]$$
- $v^k(x_i, x_i)$: expected value per click conditional on receiving a signal just high enough to win position k.

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- $v^k(x_i, x_i)$: expected value per click conditional on receiving a signal just high enough to win position k.
- **The Generalized Winner's Curse:** For all $k \in \{1, 2, \cdots, K\}$, $v^k(x_i, x_i) \leq v^{k+1}(x_i, x_i)$.

Main Results: Efficiency

One-dimensional GSP and VCG

- Each bidder i submits a bid $b_i \in \mathbb{R}$ that applies for all positions.
- Bidders receive positions in the rank ordering of bids.
- GSP: The bidder who wins k pays $\alpha_k b_{(k+1)}$.
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GSP Payment	$300 \times 8 = 2400$	$100 \times 3 = 300$	0
VCG Payment	$200 \times 8 + 100 \times 3 = 1900$	$100 \times 3 = 300$	0

Inefficiency of One-dimensional GSP and VCG

Proposition 1

Given any value function $v(x_i, x_{-i})$ satisfying assumptions **A1**-**A3**, *the GSP auction can be inefficient.*

Proposition 2

For any non-trivially interdependent value function $v(x_i, x_{-i})$ satisfying assumptions **A1**-**A3** *and $\frac{\partial v_i}{\partial x_j} \neq 0$ for $i \neq j$, the VCG auction can be inefficient.*

Sources of Inefficiency in One-dimensional Auctions

Equilibrium Condition:

$$g_1(x_i|x_i)E\left[\Pi_1 - \Pi_2 \middle| X = x_i, Y_1 = x_i\right] + g_2(x_i|x_i)E\left[\Pi_2 \middle| X = x_i, Y_2 = x_i\right] = 0$$

- Bidders are restricted to bid the same for position 1 and 2.
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  - In both GSP and VCG: $v^1(x_i, x_i) \leq v^2(x_i, x_i)$ under the Generalized Winner's Curse.
- Bid-shading incentive is stronger as x_i gets higher. The differentiated bid-shading incentives across signals leads to non-monotonicity of $\beta(x_i)$.
- Conjecture: Allowing bidders to bid differently for two positions can improve efficiency.

K-dimensional Position Auctions

- Each bidder submits K bids $(b_i^1, b_i^2, \cdots, b_i^K) \in \mathbb{R}^{\mathbb{K}}$, i.e., a bid for 1^{st} position, a bid for 2^{nd} position, etc.
- Rank all bids for the same position; Assign k to the highest bidder of k among those whose bids do not win a position better than k.
- K-D GSP: The bidder who wins k pays $\alpha_k b^k_{(k+1)}$.
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K-D GSP Payment	$300 \times 8 = 2400$	$100 \times 6 = 600$	0
K-D VCG Payment	$200 \times 8 + 100 \times 6 = 2200$	$100 \times 6 = 600$	0

Equilibria of K-dimensional GSP and VCG

Proposition 3 (BNE of K-D VCG)

The unique symmetric BNE in K-D VCG is characterized as follows: proof
For any position $k \in \{1, 2, \cdots, K\}$:

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Proposition 4 (BNE of K-D GSP)

The unique symmetric BNE in K-D GSP is characterized as follows: ▸ proof
For the last position K:

$$\beta_K(x_i) = v^K(x_i, x_i)$$

For position $k \in \{1, 2, \cdots, K-1\}$:

$$\beta_k(x_i) = v^k(x_i, x_i) - \frac{\alpha_{k+1}}{\alpha_k}\left[v^k(x_i, x_i) - \int_0^{x_i} \beta_{k+1}(y_{k+1}) dG_{k+1}\left(y_{k+1} \middle| X = x_i, Y_k = x_i\right)\right]$$

Example

Consider the K-dimensional VCG auction and K-dimensional GSP auction with $K = 2$ positions and $N = 3$ bidders, with CTR normalized to $(1, \alpha_2)$. $\alpha_2 \in [0, 1]$. x_i i.i.d. on $U[0, 1]$. v_i is given by

$$v_i = v(x_i, x_j, x_k) = \lambda x_i + \frac{1-\lambda}{2}(x_j + x_k) \quad \lambda \in \left[\frac{1}{3}, 1\right]$$

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- $\lambda = 1$: independent pure private values
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α_2 represents the relative quality of position 2 compared to position 1:

- $\alpha_2 = 1$: identical items
- $\alpha_2 = 0$: single item

Example: Equilibrium of K-D VCG with $\alpha_2 = 0.75$

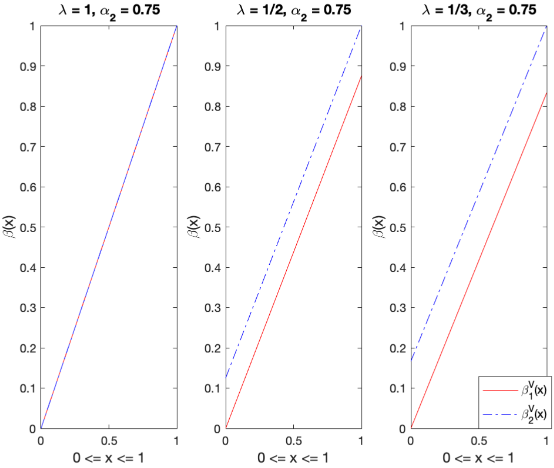

Figure 1: Equilibrium Bidding Strategies for Positions 1 and 2 in K-dimensional VCG Auction

Example: Equilibrium of K-D GSP with $\alpha_2 = 0.75$

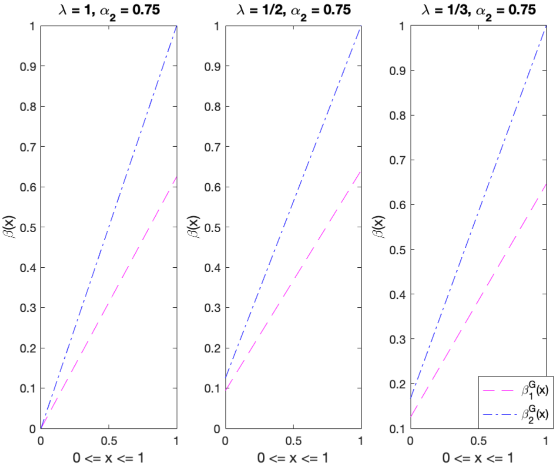

Figure 2: Equilibrium Bidding Strategies for Positions 1 and 2 in K-dimensional GSP Auction

Example: Equilibrium of K-D Auctions with $\alpha_2 = 0.75$

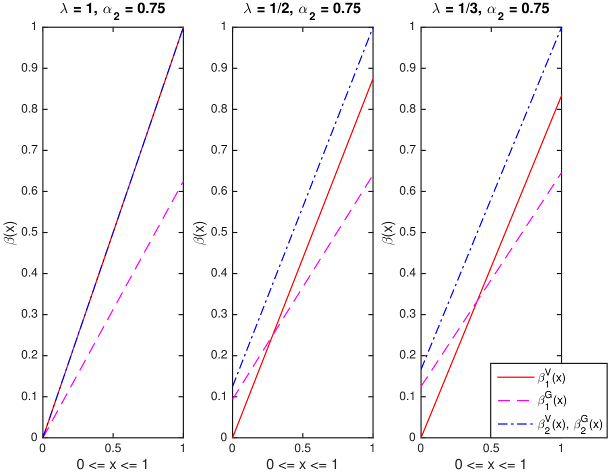

Figure 3: Equilibrium Bidding Strategies in K-dimensional VCG and GSP Auction

Example: Equilibrium of K-D Auctions with $\alpha_2 = 0.25$

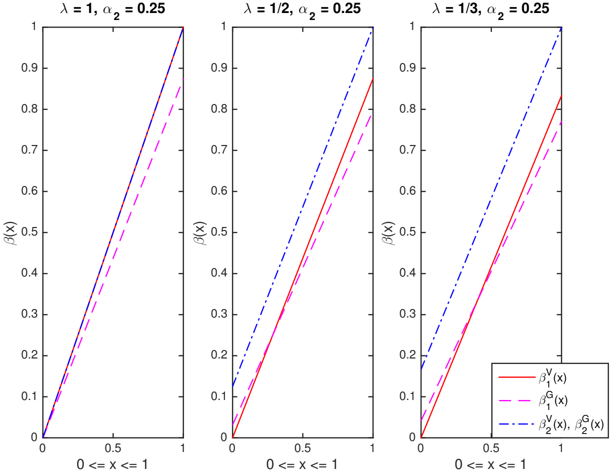

Figure 4: Equilibrium Bidding Strategies in K-dimensional VCG and GSP Auction

Generalized English Auction (GEA)

- Ascending clock showing current price; bidders drop out at any time.
- Auction ends when only one bidder is left.
- Drop-out prices: $p_N \leq p_{N-1} \leq \cdots \leq p_2$
- The remaining bidder wins Position 1 and pays $\alpha_1 \times p_2$, the last drop-out bidder wins Position 2 and pays $\alpha_2 \times p_3$, etc.

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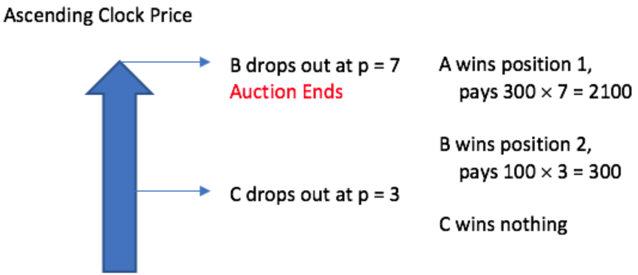
- The remaining bidder wins Position 1 and pays $\alpha_1 \times p_2$, the last drop-out bidder wins Position 2 and pays $\alpha_2 \times p_3$, etc.

Example: 3 Advertisers: A, B, and C; 2 positions: CTR=(300, 100)



Ex-post Equilibrium of GEA

Proposition 5

At any time of the auction, an active bidder's equilibrium drop-out strategy depends on the drop-out price history **AND** *the number of remaining bidders.* ▸ *proof*

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- *Fewer bidders than positions are left:* $n \leq K$
  $b_n^*(x_i | p_N, \cdots, p_{n+1}) = v^{(n-1)}(x_i, x_i, \underbrace{y_n, y_{n+1}, \cdots, y_{N-1}}_{(N-n) \text{ lowest signals}}) -$
  $\frac{\alpha_n}{\alpha_{n-1}} \left[v^{(n-1)}(x_i, x_i, \underbrace{y_n, y_{n+1} \cdots, y_{N-1}}_{(N-n) \text{ lowest signals}}) - p_{n+1} \right]$

Main Results: Revenue

Revenue Comparison

Proposition 6

For any value function $v(x_i, x_{-i})$ and distribution of signals $F(x_1, x_2, \cdots, x_N)$ that satisfy assumptions **A1-A5**, ▸ proof ▸ proof

$$R^{GEA} \geq R^{K-VCG} = R^{K-GSP}$$

Revenue Comparison

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For any value function $v(x_i, x_{-i})$ and distribution of signals $F(x_1, x_2, \cdots, x_N)$ that satisfy assumptions **A1**-**A5**, proof proof

$$R^{GEA} \geq R^{K-VCG} = R^{K-GSP}$$

Corollary 1

When bidders' signals are independently and identically distributed, for any value function $v(x_i, x_{-i})$ that satisfies **A1**-**A3**,

$$R^{GEA} = R^{K-VCG} = R^{K-GSP}$$

Characterization of the Optimal Position Auction

Proposition 7

Given a profile of bidders' signals (x_i, x_{-i}), suppose the bidders receive positions in the rank ordering of their signals under allocation rule $q^*(x_i, x_{-i})$. Suppose also that the payment rule is given by

$$p_i^*(x_i, x_{-i}) = q_i^*(x_i, x_{-i})v_i(x_i, x_{-i}) - \int_0^{x_i} q_i^*(s, x_{-i})\frac{\partial v_i(s, x_{-i})}{\partial s}ds$$

Then (q^*, p^*) is an optimal position auction among all the ex-post IC and IR mechanisms subject to no reserve price. When bidders have independent signals, this auction is optimal among all Bayesian IC and IR mechanisms. proof proof

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Then (q^*, p^*) is an optimal position auction among all the ex-post IC and IR mechanisms subject to no reserve price. When bidders have independent signals, this auction is optimal among all Bayesian IC and IR mechanisms. ▸ proof ▸ proof

Proposition 8

When bidders have independent signals, the optimal revenue can be practically implemented by GEA, K-dimensional GSP auction, and K-dimensional VCG auction. ▸ proof

Conclusions

Summary of Results

	GSP	VCG	GEA
1-dimensional	Inefficient	Inefficient	Efficient Revenue: 1^{st} $(*)$
K-dimensional	Efficient Revenue: 2^{nd} $(*)$	Efficient Revenue: 2^{nd} $(*)$	

$(*)$: Revenue equivalent under independent signals. This is also the optimal revenue subject to no reserve price.

Summary of Results

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(∗): Revenue equivalent under independent signals. This is also the optimal revenue subject to no reserve price.

Conclusions

- Allowing bidders to condition bids on positions improves efficiency and revenue.
- There is a trade-off between simplicity v.s. efficiency and revenue in auction design.

Future Research Directions

Position Auctions with Multi-unit Demands (working paper)

- Bidders may demand multiple ad slots under the same keyword.
- This paper extends the study of auction theory into vertically differentiated items with multi-unit demands.
- I propose a VCG auction and a two-stage ascending clock auction that combines the features of “Clinching” Auction in Ausubel (2004) and Generalized English Auction to allocate positions efficiently.

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Test Theoretical Results Empirically and Experimentally

- Test the efficiency and revenue properties using experimental data
- Quantify the revenue effect from adopting a multi-dimensional bidding language in GSP and VCG

Thank You!

Lemma 1: Efficiency Condition

Lemma 1

A one-dimensional position auction can be efficient if and only if there exists a symmetric and strictly monotonic equilibrium bidding strategy $\beta(x_i)$.

Lemma 2: BNE of 1-D GSP

Lemma 2

In the one-dimensional GSP auction with 2 positions, if a monotonic Bayesian equilibrium bidding strategy $\beta(x_i)$ exists, then $b^* = \beta(x_i)$ maximizes

$$\Pi(b_i|x_i) = \int_0^{\beta^{-1}(b_i)} \int_0^{y_1} \alpha_1 [v^{1,2}(x_i, y_1, y_2) - \beta(y_1)] g_i^{2,1}(y_2, y_1|x_i) dy_2 dy_1 + \int_{\beta^{-1}(b_i)}^{\bar{x}_i} \int_0^{\beta^{-1}(b_i)} \alpha_2 [v^{1,2}(x_i, y_1, y_2) - \beta(y_2)] g_i^{2,1}(y_2, y_1|x_i) dy_2 dy_1$$

Take FOC yields

For all $x_i \in [0, \bar{x}]$, $\beta(x_i)$ satisfies the Volterra equation

$$\beta(x_i) = \frac{g_1(x_i|x_i)\left[(\alpha_1 - \alpha_2) v^1(x_i, x_i) + \alpha_2 \int_0^{x_i} \beta(y_2) g_{2|1}(y_2|x_i, x_i) dy_2\right] + g_2(x_i|x_i)\alpha_2 v^2(x_i, x_i)}{\alpha_1 g_1(x_i|x_i) + \alpha_2 g_2(x_i|x_i)} \tag{10}$$

Return

Proof of Proposition 1: Inefficiency of 1-D GSP

In a one-dimensional GSP auction with two positions, the equilibrium condition can be written as

$$g_1(x_i|x_i)E\Big[\Pi_1^G - \Pi_2^G\Big|X = x_i, Y_1 = x_i\Big] + g_2(x_i|x_i)E\Big[\Pi_2^G\Big|X = x_i, Y_2 = x_i\Big] = 0$$

When $x_i \to \bar{x}$, $g_2(x_i|x_i) \to 0$, then $g_1(x_i|x_i)E\Big[\Pi_1^G - \Pi_2^G\Big|X = x_i, Y_1 = x_i\Big] = 0$. Suppose the BNE $\beta^G(x_i)$ is strictly increasing. Then

$$\begin{aligned}&\lim_{\alpha_2 \to \alpha_1} E\Big[\Pi_1^G - \Pi_2^G\Big|X = x_i, Y_1 = x_i\Big] \\ =&\alpha_1 \int_0^{x_i} \Big(\beta^G(y_2) - \beta^G(x_i)\Big) g_{2|1}(y_2|x_i, x_i) dy_2 < 0\end{aligned}$$

So there always exists (α_1, α_2) under which $FOC < 0$ around x_i close to $\bar{x}$, contradicting the assumption that $\beta^G(x_i)$ is an equilibrium.

◂ Return

Lemma 3: BNE of 1-D VCG

Lemma 3

In the one-dimensional VCG auction with 2 positions, if a monotonic Bayesian equilibrium bidding strategy $\beta(x_i)$ exists, then $b^* = \beta(x_i)$ maximizes

$$\Pi(b_i|x_i) = \int_0^{\beta^{-1}(b_i)} \int_0^{y_1} \Big\{\alpha_1[v^{1,2}(x_i, y_1, y_2) - \beta(y_1)] + \alpha_2[\beta(y_1) - \beta(y_2)]\Big\} g_i^{2,1}(y_2, y_1|x_i) dy_2 dy_1$$
$$+ \int_{\beta^{-1}(b_i)}^{\bar{x}_i} \int_0^{\beta^{-1}(b_i)} \alpha_2[v^{1,2}(x_i, y_1, y_2) - \beta(y_2)] g_i^{2,1}(y_2, y_1|x_i) dy_2 dy_1$$

The FOC implies $\beta(x_i)$ is characterized by

$$\beta(x_i) = \frac{g_1(x_i|x_i)(\alpha_1 - \alpha_2)v^1(x_i, x_i) + g_2(x_i|x_i)\alpha_2 v^2(x_i, x_i)}{g_1(x_i|x_i)(\alpha_1 - \alpha_2) + g_2(x_i|x_i)\alpha_2}$$

Return

Proof of Proposition 2: Inefficiency of 1-D VCG

$$\beta^V(x_i) = \gamma(x_i; \alpha_1, \alpha_2) v^1(x_i, x_i) + \Big(1 - \gamma(x_i; \alpha_1, \alpha_2)\Big) v^2(x_i, x_i)$$

Take derivative of $\beta(x_i) = \gamma(x_i) v^1(x_i, x_i) + (1 - \gamma(x_i)) v^2(x_i, x_i)$ with respect to x_i:

$$\frac{d\beta^V(x_i)}{dx_i} = \underbrace{\gamma(x_i)\left[\frac{\partial v^1(x_i, x_i)}{\partial x_i}\right] + (1 - \gamma(x_i))\left[\frac{\partial v^2(x_i, x_i)}{\partial x_i}\right]}_{\text{bid-increasing incentive from higher expected values}}$$

$$+ \underbrace{\frac{\partial \gamma(x_i)}{\partial x_i}\Big[v^1(x_i, x_i) - v^2(x_i, x_i)\Big]}_{\text{bid-shading incentive from the ``winner's curse''}}$$

$\frac{\partial \gamma(x_i)}{\partial x_i} \to \infty$ when $x_i \to \bar{x}$ and $\alpha_2 \to \alpha_1$, so $\frac{d\beta^V(x_i)}{dx_i}$ must be negative under some (α_1, α_2).

Return

Proof of Proposition 3

Suppose all of bidder i's opponents adopt $\beta(x)$. The FOC implies that in equilibrium, a bidder should be indifferent between position k and $k+1$ when $Y_k = x_i$:

$$E\left[\alpha_k v_i - \sum_{j=k}^{K}(\alpha_j - \alpha_{j+1})\beta_j(Y_j)\Big|X = x_i, Y_k = x_i\right]$$

$$= E\left[\alpha_{k+1} v_i - \sum_{j=k+1}^{K}(\alpha_j - \alpha_{j+1})\beta_j(Y_j)\Big|X = x_i, Y_k = x_i\right]$$

which yields

$$\alpha_k v^k(x_i, x_i) - (\alpha_k - \alpha_{k+1})\underbrace{E[\beta_k(Y_k)|X = x_i, Y_k = x_i]}_{\beta_k(x_i)} = \alpha_{k+1} v^k(x_i, x_i)$$

$$E[\beta_k(Y_k)|X = x_i, Y_k = x_i] = \beta_k(x_i) = v^k(x_i, x_i)$$

Therefore, the equilibrium bidding strategy is given by

$$b_i^{k*} = \beta_k(x_i) = v^k(x_i, x_i)$$

Return

Proof of Proposition 4

Suppose all of bidder i's opponents adopt $\beta(x)$. The FOC of i's objective function implies that in equilibrium, a bidder should be indifferent between position k and $k+1$ when $Y_k = x_i$:

$$E\left[\alpha_k(v_i - \beta_k(Y_k))\middle| X = x_i, Y_k = x_i\right] = E\left[\alpha_{k+1}(v_i - \beta_{k+1}(Y_{k+1}))\middle| X = x_i, Y_k = x_i\right]$$

which yields

$$\alpha_k\Big(v^k(x_i,x_i) - \underbrace{E[\beta_k(Y_k)|X = x_i, Y_k = x_i]}_{\beta_k(x_i)}\Big)$$
$$= \alpha_{k+1}\Big(v^k(x_i,x_i) - E[\beta_{k+1}(Y_{k+1})|X = x_i, Y_k = x_i]\Big)$$

Therefore, the equilibrium bidding strategy is given by

$$b_i^{k*} = \beta_k(x_i) = v^k(x_i,x_i) - \frac{\alpha_{k+1}}{\alpha_k}[v^k(x_i,x_i) - E[\beta_{k+1}(Y_{k+1})|X = x_i, Y_k = x_i]]$$

◂ Return

Proof of Proposition 5

- When all N bidders are "in", suppose all the opposing bidders adopt strategy b_N^*, bidder i will not drop out until the expected payoff from the last position K falls below zero.
- i wins position K by dropping out at p only if $(N-K)$ lowest signal bidders drop out simultaneously, which implies $Y_K = Y_{K+1} = \cdots = Y_{N-1} = y_K$. i's expected payoff is

$$\alpha_K v^{(K)}(x_i, y_K, \cdots, y_K) - \alpha_K v^{(K)}(y_K, y_K, \cdots, y_K) \geq 0 \quad \textit{iff} \quad x_i \geq y_K$$

  So bidder i's optimal drop-out price is $p = v^{(K)}(x_i, x_i, \cdots, x_i)$.
- When $(N-n)$ bidders have dropped out, but $n \geq K+1$ bidders are still in the auction, we just need to replace the lowest $(N-n)$ signals by the revealed signals. i's optimal drop-out price is

$$v^{(K)}(x_i, \underbrace{x_i, \cdots, x_i}_{(n-K)}, \underbrace{y_n, y_{n+1}, \cdots, y_{N-1}}_{(N-n)\text{ lowest signals}})$$

Proof of Proposition 5

- When only $n \leq K$ bidders left in the auction, a bidder should be indifferent between getting the current lowest position n at price p_{n+1} and an upper position $(n-1)$ at a higher price b in equilibrium.
- The lowest value remaining opposing bidder with signal y_{n-1} drops out at b defined by b_n^*:

$$b = v^{(n-1)}(y_{n-1}, y_{n-1}, \cdots, y_N) - \frac{\alpha_n}{\alpha_{n-1}}\left[v^{(n-1)}(y_{n-1}, y_{n-1}, \cdots, y_N) - p_{n+1}\right]$$

- The expected payoff from winning $(n-1)$ is $\Pi_{n-1} = \alpha_{n-1}\left[v^{(n-1)}(x_i, y_{n-1}, y_n, \cdots, y_N) - b\right]$.
- The expected payoff from winning n is $\Pi_n = \alpha_n\left[v^{(n-1)}(x_i, y_{n-1}, y_n \cdots, y_N) - p_{n+1}\right]$.
- $\Pi_{n-1} - \Pi_n \geq 0$ if and only if $(\alpha_{n-1} - \alpha_n)\left[v^{(n-1)}(x_i, y_{n-1}, y_n \cdots, y_N) - v^{(n-1)}(y_{n-1}, y_{n-1}, y_n \cdots, y_N)\right] \geq 0$ So b_n^* is best response bid for i when $n \leq K$ given all opponents adopt b^*.

Return

Proof of Proposition 6: $R^E \geq R^V$

For the last position K, the expected prices in GEA and K-dimensional VCG are given by

$$E[p^{E,(K)}] = E[v^{(K)}(Y_K, Y_K; Y_{K+1}, Y_{K+2}, \cdots, Y_{N-1})|\{Y_{K-1} > X > Y_K\}]$$
$$E[p^{V,(K)}] = E[v^K(Y_K, Y_K)|\{Y_{K-1} > X > Y_K\}]$$

For any position $k \in [1, K-1]$, the expected prices are given by

$$E[p^{E,(k)} - p^{E,(k+1)}] = (\alpha_k - \alpha_{k+1})E[v^{(k)}(Y_k, Y_k; Y_{k+1}, .., Y_{N-1})|\{Y_{k-1} > X > Y_k\}]$$
$$E[p^{V,(k)} - p^{V,(k+1)}] = (\alpha_k - \alpha_{k+1})E[v^k(Y_k, Y_k)|\{Y_{k-1} > X > Y_k\}]$$

Apply Linkage Principle twice gives $E[p^{E,(k)}] \geq E[p^{V,(k)}]$ for all k. Return

Proof of Proposition 6: $R^V = R^G$ (Method 1)

For the last position K, the expected prices in K-dimensional VCG and GSP are given by

$$E[p^{V,(K)}] = \alpha_K E[v^K(Y_K, Y_K)|\{Y_{K-1} > X > Y_K\}]$$
$$E[p^{G,(K)}] = \alpha_K E[v^K(Y_K, Y_K)|\{Y_{K-1} > X > Y_K\}]$$

For any position $k \in [1, K-1]$, the expected prices are given by

$$\begin{aligned} E[p^{V,(k)}] &= (\alpha_k - \alpha_{k+1})E[\beta_k^V(Y_k)|\{Y_{k-1} > X > Y_k\}] + E[p^{V,(k+1)}] \\ &= (\alpha_k - \alpha_{k+1})E[v^k(Y_k, Y_k)|\{Y_{k-1} > X > Y_k\}] + E[p^{V,(k+1)}] \end{aligned}$$

$$\begin{aligned} E[p^{G,(k)}] &= \alpha_k E[\beta_k^G(Y_k)|\{Y_{k-1} > X > Y_k\}] \\ &= \alpha_k E\big[v^k(Y_k, Y_k) - [\frac{\alpha_{k+1}}{\alpha_k} v^k(Y_k, Y_k) - E[\beta_{k+1}^G(Y_{k+1})]]|\{Y_{k-1} > X > Y_k\}\big] \\ &= (\alpha_k - \alpha_{k+1})E[v^k(Y_k, Y_k)|\{Y_{k-1} > X > Y_k\}] + E[p^{G,(k+1)}] \end{aligned}$$

Therefore, $E[p^{V,(k)}] = E[p^{G,(k)}]$ for all k.

Return

Proof of Proposition 6: $R^V = R^G$ (Method 2)

With $K = 2$ positions, the expected payment of a bidder with signal x_i in K-D VCG and GSP are given by

$$m^V(x_i) = Pr(x_i \geq Y_1) E\Big[(\alpha_1 - \alpha_2) \underbrace{v^1(Y_1, Y_1)}_{\beta_1^V(Y_1)} + \alpha_2 \underbrace{v^2(Y_2, Y_2)}_{\beta_2^V(Y_2)} \Big| x_i \geq Y_1\Big]$$

$$+ Pr(Y_2 \leq x_i < Y_1) E\Big[\alpha_2 \underbrace{v^2(Y_2, Y_2)}_{\beta_2^V(Y_2)} \Big| Y_2 \leq x_i < Y_1\Big]$$

$$m^G(x_i) = Pr(x_i \geq Y_1) E\Big[\alpha_1 \underbrace{\left\{v^1(Y_1, Y_1) - \frac{\alpha_2}{\alpha_1} v^1(Y_1, Y_1) + \frac{\alpha_2}{\alpha_1} E[v^2(Y_2, Y_2)|Y_1]\right\}}_{\beta_1^G(Y_1)} \Big| x_i \geq Y_1\Big]$$

$$+ Pr(Y_2 \leq x_i < Y_1) E\Big[\alpha_2 \underbrace{v^2(Y_2, Y_2)}_{\beta_2^G(Y_2)} \Big| Y_2 \leq x_i < Y_1\Big]$$

According to the Law of Iterated Expectations,

$$E\Big[E[v^2(Y_2, Y_2)|Y_1] \Big| Y_1 \leq x_i\Big] = E[v^2(Y_2, Y_2)|Y_1 \leq x_i]$$

So $m^V(x_i) = m^G(x_i)$. Similar argument applies for any $K \geq 2$.

Return

Proof of Proposition 7

Return

Lemma 4

A position auction mechanism (q, p) is ex post IC and IR if and only if for all i and (x_i, x_{-i}), $q_i(x_i, x_{-i})$ is weakly increasing in x_i, and

$$u_i(x_i, x_{-i}) = u_i(0, x_{-i}) + \int_0^{x_i} \left[\frac{\partial v_i(s, x_{-i})}{\partial s}\right] q_i(s, x_{-i}) ds \quad \text{for all} \quad x_{-i}$$

$$u_i(0, x_{-i}) \geq 0 \quad \text{for all} \quad x_{-i}$$

Lemma 5

In any ex post IC and IR mechanism, the ex ante expected revenue is given by

$$ER = \int_x \sum_i \left\{ q_i(x_i, x_{-i}) \left\{ v_i(x_i, x_{-i}) - \frac{1 - F_i(x_i|x_{-i})}{f_i(x_i|x_{-i})} \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \right\} \right\} f(x) dx$$
$$- \int_{x_{-i}} \sum_i u_i(0, x_{-i}) f_{-i|0}(x_{-i}|0) dx_{-i}$$

Proof of Proposition 7

Lemma 6

A position auction mechanism (q, p) is Bayesian IC and IR if for every i, for any report x, the expected CTR $q_i(x_i, x_{-i})$ is weakly increasing in x_i, and

$$U_i(x_i) = U_i(0) + \int_{x_{-i}} \int_0^{x_i} \Big[\frac{\partial v_i(s, x_{-i})}{\partial s}\Big] q_i(s, x_{-i}) ds f_{-i}(x_{-i}) dx_{-i}$$

$$U_i(0) \geq 0$$

Lemma 7

For any Bayesian IC and IR mechanism that satisfy the conditions in lemma 6, the ex ante expected revenue is given by

$$ER = \int_x \sum_i \Big\{ q_i(x_i, x_{-i}) \Big\{ v_i(x_i, x_{-i}) - \frac{1 - F_i(x_i)}{f_i(x_i)} \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \Big\} \Big\} f(x) dx - \sum_i U_i(0)$$

Return

Proof of Proposition 8

- Substitute $\hat{x}^k(x_{-i}) = \hat{X}^k(x_{-i})$ into the optimal auction (q^*, p^*) defined in Proposition 7, it is trivial that $q^* = q^V$.
- Substitute the allocation rule $q^V = q^*$ into the payment rule

$$p_i^*(x_i, x_{-i}) = q_i^*(x_i, x_{-i})v_i(x_i, x_{-i}) - \int_0^{x_i} q_i^*(s, x_{-i})\frac{\partial v_i(s, x_{-i})}{\partial s}ds$$

  It can be shown that $p_i^* = p_i^V$. So (q^*, p^*) is equivalent to (q^V, p^V) under regularity condition **R3**.
- The payment of each bidder depends on the entire signal profile in the Generalized-VCG, while it depends only on a subset of bidders' signals in GEA and depends only on each bidder's own signal in K-D GSP and K-D VCG. $R^{Optimal} \geq R^{GEA} \geq R^{K-VCG} = R^{K-GSP}$ under affiliated signals by Linkage Principle.

◂ Return