Position Auctions with Interdependent Values *

Haomin Yan[‡]

March 22, 2019

Abstract

This paper extends the theoretical study of position auctions to an interdependent values model in which each bidder's value depends on its opponents' information as well as its own information. Position auctions are used by major search engines to allocate multiple advertising positions on search result pages. In this paper, I examine efficiency and revenues of three position auction formats: Generalized Second Price (GSP) auctions, VCG-like auctions, and Generalized English Auctions (GEA). I find that both the GSP auction and the VCG-like auction with one-dimensional bidding language can be inefficient under interdependent values, which contrasts previous literature that favors the GSP auction for its simplicity. I next show this inefficiency problem can be fully resolved by adopting a multi-dimensional bidding language that allows bidders to bid differently across positions. Moreover, the dynamic GEA that implicitly adopts a multi-dimensional bidding language always implements efficiency in an ex-post equilibrium. Then I provide a revenue ranking of the three efficient position auctions and characterize the optimal position auction subject to no reserve price under interdependent values. I find that under independent signals and a set of regularity conditions, the three efficient position auctions also implement the optimal revenue subject to no reserve price.

JEL Classification: D44, D47, D61, D82

Keywords: position auctions; interdependent values; generalized second price auction; VCG auction; generalized English auction

---

*I am very grateful to Professor Lawrence Ausubel, Professor Daniel Vincent and Professor Emel Filiz-Ozbay for continual support and many helpful suggestions through the development of this research. I also thank conference and seminar participants for their helpful comments. This research was supported by funding under graduate assistantship at University of Maryland.

[†]University of Maryland, College Park, MD 20740, United States. Email: yanhaomin@gmail.com

[‡]Current Affiliation: FTI Consulting, Boston, MA 02114, United States.

1 Introduction

Position auctions are used by search engines such as Google and Yahoo! to allocate sponsored advertising slots to advertisers on search result pages. When an Internet user enters a keyword or phrase on a search engine, the list of advertisements generated by that search is the result of a position auction. Because of consumers' sequential search habits[1], advertising links placed on the top of web page receive more clicks than those placed on the bottom of web page, representing a typical set of vertically differentiated items. Each advertising link's click probability can be measured by click-through-rate (CTR), which is given by the average number of clicks the link receives per unit time. There are three different position auction formats that have been analyzed in the literature, including the Generalized Second Price (GSP) auction, the Vickrey-Clark-Groves (VCG) auction, and the Generalized English Auction (Edelman et al. 2007[15]). Variants of the Generalized Second Price auction have been practically adopted by search engines[2]. In the standard model of GSP auction[3] with per-click payment rule, advertisers submit one-dimensional per-click bids that can be applied to any position. The positions are allocated according to the ranking of bids, and each bidder who wins a position pays the bid of the bidder who is placed one position below for each click. Previous literature has shown that both the GSP auction and the VCG auction always implement the efficient allocation in equilibrium under complete information (Edelman et al. 2007[15]; Varian 2007[38]). The GSP auction is especially favored for its simple design: one-dimensional bids are used to determine the allocation

[1]Consumers tend to search from top to bottom when reading a list and may end search at any time, so the top links are more likely to be clicked than the bottom links. This search behavior can be viewed as a rule of thumb, or as a rational behavior given positive search cost and correct expectation about advertisers' relevance (Athey and Ellison 2009[4]; Chen and He 2011[11]).

[2]The Generalized Second Price auction has several variations in its form. One important variation is to adopt a vector of "quality scores" computed based on click-through-rate history to adjust bids and rank advertisers in the order of adjusted bids instead of raw bids. Another variation is to adopt a pay-per-impression scheme instead of a pay-per-click scheme. Under the pay-per-click scheme, an advertiser will be charged every time a user clicks on its advertisement. Under the pay-per-impression scheme, an advertiser will be charged every time a user sees the search result page that contains its advertisement regardless of whether the user clicks on the advertisement or not. Google currently uses the pay-per-click GSP auction with quality scores.

[3]Following Edelman et al. (2007)[15], the GSP auction in this paper refers to the GSP auction with pay-per-click payment rule and leaves aside the quality scores.

of multiple positions, and payment for each position depends only on the highest losing bid for that specific position.

The motivation of this paper comes from that previous literature has modeled position auctions either under complete information or under incomplete information with independent private values (Gomes and Sweeney 2014[21]). However, given the fact that advertisers bidding for the same keyword are likely to be oligopoly competitors operating in the same market subject to common aggregate demand shocks, neither the complete information model nor the independent private values model is sufficient for capturing both (1) the uncertainty in sponsored search markets and (2) the oligopoly relationship among advertisers in position auctions.

The justification of complete information in the literature comes from the claim that advertisers learn about their own values as well as their rivals' values from information revealed in previous auction rounds. This claim implicitly assumes that advertisers' values do not evolve over time intervals between bidding, so the information revealed in previous rounds fully reveals advertisers' values for the current round. In practice, considerable uncertainty exists in sponsored search auctions. For example, consider the keyword "iphone." Each advertiser's value from receiving a click of the online advertisement depends on how likely consumers are going to purchase a new iphone upon click, which can be affected significantly by product upgrading and new releases in the iphone market. Consider the time when Apple Inc. releases a new version of iphone, then each advertiser's value changes continuously over time after the first day of release, and it is not practical to precisely predict consumer demand or advertisers' values in advance. Consider another search phrase for example, "hotels in New York City," then each advertiser's value per click depends on how likely consumers are going to book a hotel after clicking on its advertisement, which can be affected by a variety of factors including weather, day of the week, time of the year, special events in New York, etc. Therefore, advertisers' values evolve continuously for many keywords related to markets with frequent demand shocks. The evolution in advertisers' values as result of shocks is also pointed out in Fershtman and Pavan (2016)[18] and Abhishek and

Hosanagar (2012)[1]. On the other hand, it is not practical for advertisers to update bids in a continuous manner given the fact that each advertiser is interested in a large set of keywords. The fluctuation in consumer demand and the existence of time interval between bid updating imply that information revealed from previous auctions does not always provide complete information about an advertiser's own value or its opponents values at the time of bidding.

Another important feature of position auction is that advertisers bidding for a slot under the same keyword are often oligopoly competitors operating in the same industry. Compared to traditional advertising, a main advantage of sponsored advertising is that it allows advertisers to effectively target consumers. This advantage naturally comes with the fact that advertisers under the same keyword are selling identical or imperfectly substitutable products or services in the market related to the search keyword. Since each advertiser's value per click evolves continuously under demand shocks, it is reasonable to assume that there exists some unknown common component in bidders' ex-post values that is driven by demand shocks in the same market. For example, when Apple Inc. releases a new version of iphone, consumers are more likely to buy a new iphone upon click, and all advertisers are subject to the same demand shock. While consumer demand cannot be precisely predicted, each advertiser can still have some imprecise estimation of its value per click. Suppose a given advertiser receives a private signal that contains information about how likely consumers are going to purchase a new iphone after the release. Then the private signals of other advertisers would be informative about the first advertiser's ex-post value per click, given the fact that consumer demand drives a common component in all advertisers' ex-post values. Since the advertisers' private signals contain information about consumer demand in the same market, it is also reasonable to assume the signals are affiliated in distribution. Therefore, the information structure in position auctions is better described by the interdependent values model introduced by Milgrom and Weber (1982)[31], in which one bidder's value can depend on other bidders' private information, and bidders' private information are affiliated. However, the performance of the GSP auction, the VCG auction and the Generalized English Auction is not well-understood when bidders have interdependent values. This paper fills the gap in the literature and extends the study of position auctions into this broader class of information structure.

In this paper, I model a single-round position auction under a symmetric interdependent values model. Since each bidder's ex-post value depends on its opponents' private information, bidders can be uncertain about their ex-post values at the time of bidding, and a generalized version of the "winner's curse" in Milgrom and Weber (1982)[31] is present: the expected value per click conditional on winning a higher ranked position is lower than that conditional on winning a lower ranked position. Winning a top position conveys some bad news, as it implies overestimation of ex-post value from receiving a click. The main analysis of this paper explores (1) how the incomplete information and the presence of the generalized "winner's curse" under interdependent values affect efficiency and revenue of GSP auctions, VCG-like auctions[4], and Generalized English Auctions, (2) how to design alternative practical auction mechanisms to improve efficiency, and (3) how the expected revenues of different auction mechanisms compare to each other and to the optimal revenue implementable subject to no reserve price.

I first show that both the GSP auction and the VCG-like auction can be inefficient under interdependent values, which contrasts with previous literature that favors the GSP auction for its simplicity. Then I propose a modification of the GSP auction and the VCG-like auction by adopting a multi-dimensional bidding language that allows each bidder to bid differently across positions. I call these two modified auctions K-dimensional GSP auctions and K-dimensional VCG auctions, respectively. I characterize the unique symmetric Bayesian Nash equilibrium in these two modified auctions and show that efficiency can be fully implemented in both auctions after adopting this multi-dimensional bidding language. On the other hand, the Generalized English Auction that implicitly adopts a multi-dimensional bidding language always implements the efficient allocation in an ex-post equilibrium. Moreover, the K-dimensional GSP

[4]Although the VCG mechanism is not defined under interdependent values, this paper studies a VCG-like auction called the *one-dimensional VCG auction* that adopts a VCG-like payment rule under interdependent values. This one-dimensional VCG auction is analogous to the second-price auction in Milgrom and Weber (1982)'s study of single-unit auctions under interdependent values.

auction and the K-dimensional VCG auction are always revenue equivalent, while the dynamic Generalized English Auction yields higher revenue under affiliated signals. In the special case of independent signals, all three efficient auctions are revenue equivalent. I also characterize the optimal position auction that generates the highest expected revenue subject to no reserve price as a direct revelation mechanism and show that under certain regularity conditions and independent signals, the K-dimensional GSP auction, the K-dimensional VCG auction, and the Generalized English Auction can implement the optimal revenue subject to no reserve price.

The inefficiency of the GSP auction and the VCG-like auction comes from the fact that both auctions use a simple one-dimensional bidding language that restricts bidders to submit the same bid for all positions, while the expected payoff of winning a higher ranked position can be lower than that of a lower ranked position, leading to bid-shading incentives in both auctions. In the VCG-like auction, this phenomenon arises solely from the presence of the "winner's curse" under interdependent values. In the GSP auction, this phenomenon arises from both the GSP payment rule and the "winner's curse." The former is pointed out by Gomes and Sweeney (2014)[21], who show that the GSP auction can be inefficient under independent private values. The intuition is that when the CTR of a lower ranked position is close to the CTR of a higher ranked position, a bidder will receive similar number of clicks from winning either position, but will pay a much higher price for each click if it wins the higher ranked position given the GSP payment rule. Upon the introduction of interdependent values, this bid-shading incentive is amplified under the "winner's curse," as winning a higher ranked position not only implies a higher price per click but also implies a lower expected value for every click. Given the presence of bid-shading incentive in both auctions, efficiency can be retained only if all bidders have the same degree of bid-shading incentives. However, this is not the case as long as there are more than one positions. With multiple differentiated positions, the bid-shading incentive is stronger for bidders with higher signals, since bidders with higher signals are more likely to win the highest position in any monotonic equilibrium, while bidders with lower signals are more concerned with winning any position instead of winning a top position. It is the differentiated bid-shading incentives across bidders that drive the inefficiency in both GSP auctions and VCG-like auctions.

The source of inefficiency in the GSP auction and the VCG-like auction implies that restricting bidders to submit the same bid for all positions can hurt efficiency. By allowing bidders to express willingness to pay separately for each position, bidders can easily incorporate the difference in expected payoffs from winning different positions into their bids. The differentiated bid-shading incentives across bidders are replaced by each bidder's differentiated bid-shading incentives across positions. This explains the efficiency of K-dimensional GSP and VCG auctions. Similarly, in the Generalized English Auction, bidders are able to update beliefs about which position they are going to win by dropping out at the current clock price during the dynamic process. The Generalized English Auction implicitly adopts a multi-dimensional bidding language, which is the main force that drives its efficiency. The revenue ranking between the Generalized English Auction and the K-dimensional VCG auction resembles the revenue ranking between the English auction and the second-price auction in Milgrom and Weber (1982)[31] and comes from the fact that dynamic auctions outperform static auctions in revenue by eliciting more information about bidders' signals through the drop-out process. On the other hand, the revenue equivalence between the K-dimensional GSP and the K-dimensional VCG auctions comes from the fact that GSP and VCG can be viewed as two different generalizations of the second-price auction under the context of position auctions: both auctions use variations of a "second-price" payment rule in which a given bidder's bid only affects its allocation but not its payment. Bidders are able to incorporate the difference in payment rules into their bidding strategies, which drives the revenue equivalence result between these two static auctions under the general assumption of affiliated signals. The revenue equivalence of all three auctions under independent signals is consistent with the well-known revenue equivalence theorem.

Comparing across the three auction formats, the Generalized English Auction has two advantages over the static GSP and VCG-like auctions under interdependent values because of its dynamic nature. First, it naturally allows bidders to update strategy on which position they are bidding for as rivals drop out in the dynamic process. This advantage yields efficiency. Second, it allows bidders to update their belief on the ex-post value per click from observing the drop-out prices of rivals who drop out before them. This advantage yields higher revenue. However, the dynamic nature of the Generalized English Auction also makes it impossible to be implemented in practice, as sponsored search auctions take place in a continuous manner in real time. It is impractical to gather all bidders to participate in a centralized clock auction at the same time. Therefore, the Generalized English Auction can be viewed as a theoretical modeling tool and a comparison benchmark rather than a practical auction format. This paper establishes that, by adopting a multi-dimensional bidding language, both the GSP auction and the VCG-like auction can achieve the same level of efficiency as in the Generalized English Auction and also yield the same level of revenue under independent signals, without losing feasibility for practical implementation.

The main contribution of this paper is the follows. First, I extend the study of position auctions into interdependent values and show that existing commonly-used position auctions, including both GSP and VCG auctions, are no longer efficient under this information structure. Second, I identify that the source of this inefficiency comes from the oversimplified bidding language and propose two alternative practical efficient auction formats based on modification of GSP and VCG auctions. Third, I provide a comparison across three different position auction formats in both efficiency and revenue and provide a discussion on the optimal auction under this broad class of information structure. The K-dimensional GSP auction and the K-dimensional VCG auction proposed in this paper have the potential for practical implementation by search engines as well as a wide range of two-sided platforms, such as Facebook, Amazon, and Yelp. The main results of this paper imply that there is a trade-off between simplicity versus efficiency and revenue in auction design: simplicity can come at a loss of efficiency and revenue. This trade-off depends critically on the information structure.

2 Related Literature

The earliest papers on position auction model it as a static game under complete information, including Edelman et al. (2007)[15] and Varian (2007)[38]. Edelman et al. (2007)[15] characterize the set of locally-envy free equilibria of the GSP auction under complete information and show that the GSP auction has a locally-envy free equilibrium that yields the same payoff outcome as the dominant strategy equilibrium of the VCG auction. Moreover, this equilibrium gives the bidder-optimal payoff among all locally-envy free equilibria. In a complementary article, Varian (2007)[38] characterizes the entire set of Nash equilibria in the GSP auction under complete information. Milgrom (2010)[30] shows that the GSP auction can be viewed as a simplified mechanism that restricts each bidder to submit the same bid for all positions. This simplification in bidding language eliminates the lowest revenue equilibrium and leaves only higher revenue equilibria under complete information. Dutting et al. (2011)[13] point out that Milgrom (2010)[30]'s result depends critically on the complete information assumption. This paper supports Dutting et al. (2011)[13]'s discussion on the trade-off between simplicity and expressiveness in mechanism design by showing that the GSP auction with one-dimensional bidding language can be sub-optimal under interdependent values, in sharp contrast to the results in Edelman et al. (2007)[15], Varian (2007)[38] and Milgrom (2010)[30] that favor the GSP auction for its simplicity and desirable properties under complete information[5].

In an incomplete information setting, Edelman et al. (2007)[15] model an ascending auction called the Generalized English Auction (GEA) that implements the same payoff outcome as the dominant strategy equilibrium of the VCG auction under independent private values. Little was known about equilibria of the GSP auction under incomplete information until Gomes and Sweeney (2014)[21] first characterized the Bayesian Nash

[5] The cost of conciseness in the design of GSP auction is also pointed out in the computer science literature. Abrams et al. (2007)[2] show that an equilibrium can fail to exist in the GSP auction when each bidder has a vector of different values for obtaining different slots. Benisch et al. (2008)[8] show that the GSP auction can be arbitrarily inefficient under some distributions of the advertisers' preferences when advertisers have private information. This paper complements these computer science studies by providing some insights on the trade-off between simplicity and efficiency from an economic perspective.

Equilibrium of the GSP auction in an independent private values model and showed this unique equilibrium can be inefficient under some click-through rate profiles. This paper is closely related to Gomes and Sweeney (2014)[21] and extends their study in three ways. First, this paper introduces informational interdependency among bidders' values while nesting the independent values model as a special case and shows that the introduction of interdependent values amplifies the inefficiency of the GSP auction. Second, this paper identifies the source of inefficiency in the GSP auction and proposes a modified GSP auction to improve efficiency. Third, this paper also compares the performance of the GSP auction to other position auction formats. Moreover, one implication of Gomes and Sweeney (2014)[21] is that the oversimplified payment rule of the GSP auction can hurt efficiency under incomplete information. This paper adds some new insights to this implication by showing that the oversimplified bidding language is another cause of inefficiency under interdependent values.

This paper is also related to the literature on auctions and mechanism design under interdependent values, most of which focus on single-unit auctions or multi-unit auctions with identical items. Milgrom and Weber (1982)[31] characterize the equilibria of second-price auctions, first-price auctions and English auctions and compare the expected revenues of these auctions under symmetric interdependent values. A number of other articles examine the existence and design of efficient mechanisms under interdependent values (Jehiel and Moldovanu, 2001[24]; Dasgupta and Maskin, 2000[12]; Perry and Reny, 2002[33]; Ausubel, 1999[5]; Ausubel and Cramton, 2004[7]). This paper extends the literature on auction design under interdependent values into multi-unit auctions with vertically differentiated items.

This paper complements the recent position auctions literature[6] that introduces some realistic assumptions into Edelman et al. (2007)[15]'s model. Some studies endogenize advertisers' values by incorporating consumer search into the model and show that firms are ranked in the order of relevance and consumers search sequentially in equilibrium (Athey and Ellison, 2011[4]; Chen and He, 2011[11]; Kominers, 2009[26]).

[6] Most recent advances in this literature are summarized in Qin et al. (2015)[34].

Several other studies introduce allocative externalities among bidders by allowing click-through rate of each position to depend on the allocation of advertisers[7] (Deng and Yu, 2009[14]; Farboodi and Jafaian, 2013[17]; Hummel and McAfee, 2014[22]; Izmalkov et al., 2016[23]; Lu and Riis, 2016[28]). There are also studies that quantify the efficiency loss that may arise in the GSP auction under different modeling assumptions, including correlated private values, allocative externalities, uncertain click-through rate profiles, etc. (Lucier and Leme, 2011[29]; Roughgarden and Tardos, 2015[35]; Caragiannis et al., 2015[9]). This paper differs from the aforementioned studies by keeping Edelman et al. (2007)[15]'s assumption of exogenous click-through rates while introducing informational interdependency in bidders' values, which to my knowledge has not been done by previous studies.

Finally, this study is related to the strand of literature on optimal mechanism design. Myerson (1981)[32] characterizes the optimal mechanism for single-unit auctions with independent private values. Ausubel and Cramton (1999)[6] find that in auction markets with perfect resale, it is optimal to allocate items efficiently. Edelman and Schwarz (2010)[16] generalize Myerson (1981)[32]'s optimal mechanism design to position auctions with independent private values and show that this optimal revenue can be implemented by a Generalized English Auction with an optimal reserve price. Roughgarden and Talgam-Cohen (2013)[36] and Li (2016)[**?**] extend the characterization of optimal single-unit auction to interdependent values. Ulku (2013)[37] characterize the optimal mechanism for allocating a set of heterogeneous items under interdependent values. The last part of this paper provides a corollary of Ulku (2013)[37] under the special environment of position auctions.

[7] There is a similar line of research in the computer science literature (Aggwaral et al., 2008[3]; Constantin et al., 2011[10]; Ghosh and Mahdian, 2008[19]; Kempe and Mahdian, 2008[25]).

3 Model

A search engine wishes to sell K positions to $N > K$ bidders[8], each with single-unit demand for an advertising position on the search result page of the same keyword. Bidders are indexed by $i \in \{1, 2, \cdots, N\}$. Positions are indexed by $k \in \{1, 2, \cdots, K\}$ according to their ranks on the web page and are vertically differentiated in their commonly known qualities measured by click-through rates (CTR): $(\alpha_1, \alpha_2, \cdots, \alpha_K)$[9], in which $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_K$. Each bidder i receives a private signal $x_i \in [0, \bar{x}]$ that affects her value from getting a click of her advertisement. The signals are distributed over $[0, \bar{x}]^N$ according to a commonly-known joint distribution function $F(x_1, x_2, \cdots, x_N)$, with density $f(x_1, x_2, \cdots, x_N)$. The value per click[10] of each bidder i depends on her private signal x_i as well as her opponents' signals $x_{-i} \in [0, \bar{x}]^{N-1}$. For any bidder i, there is a function $v_i : [0, \bar{x}]^N \to \mathbb{R}$ that maps a signal profile (x_i, x_{-i}) to bidder i's ex-post value per click, $v_i(x_i, x_{-i})$.

For every bidder i, $v_i(x_i, x_{-i})$ satisfies the following assumptions[11]:

A1 (*Value Symmetry*): The value function $v_i(x_i, x_{-i})$ is symmetric across bidders. Moreover, the function $v_i(x_i, x_{-i})$ is symmetric in its last $N-1$ arguments, which implies that each bidder's value $v_i(x_i, x_{-i})$ is preserved under any permutation of opponents' signals x_{-i}.

A2 (*Value Monotonicity*): For any bidder i, $v_i(x_i, x_{-i})$ is nonnegative, continuous and strictly increasing in x_i, non-decreasing in every component of x_{-i}:

$$\frac{\partial v_i(x_i, x_{-i})}{\partial x_i} > 0, \quad \frac{\partial v_i(x_i, x_{-i})}{\partial x_j} \geq 0, \quad \forall j \neq i \tag{1}$$

[8] In this paper, I use masculine pronoun for the auctioneer (search engine) and feminine pronouns for the bidders (advertisers).

[9] Following Edelman et al. (2007)[15], the CTR of position k is measured by the expected number of clicks per period received by the advertiser whose advertisement is placed on position k. The CTR of each position does not depend on the identity of bidder placed on that position or any other position.

[10] Following Edelman et al. (2007)[15], I assume each bidder's value from getting a click does not depend on the position of her advertisement.

[11] Assumptions A1-A5 follow from Milgrom and Weber (1982)[31].

Bidders have non-trivially interdependent values if the second inequality is strict, $\frac{\partial v_i(x_i,x_{-i})}{\partial x_j} > 0$. When the second inequality is not strict, the model breaks down into a pure private values model.

A3 (*Single-crossing Condition*): For all i, for all $j \neq i$, for all signals $(x_1, x_2, \cdots, x_N)$,

$$\frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \geq \frac{\partial v_j(x_j, x_{-j})}{\partial x_i} \tag{2}$$

The single-crossing condition is a standard assumption in the literature on mechanism design under interdependent values. It implies that if bidder i has a higher value than bidder j at signal profile (x_i, x_{-i}), then bidder i must still have a higher value than bidder j at signal profile $(x_i^{'}, x_{-i})$ where $x_i^{'} > x_i$. This is a necessary condition for existence of efficient mechanisms. Assumptions **A1**-**A3** also imply that the ranking of signals is aligned with the ranking of bidders' ex-post values. The bidder who receives the k-th highest signal also has the k-th highest ex-post value.

I assume the joint density function $f(x_1, x_2, \cdots, x_N)$ satisfies the following assumptions:

A4 (*Signal Symmetry*): $f(x_1, x_2, \cdots, x_N)$ is a symmetric function of its arguments.

A5 (*Signal Affiliation*): The variables $x_1, x_2, \cdots, x_N$ are affiliated. For all $x, x' \in [0, \bar{x}]^N$,

$$f(x \vee x')f(x \wedge x') \geq f(x)f(x') \tag{3}$$

I restrict attention to symmetric equilibria in this paper[12]. Given symmetry of the model, it suffices to study the equilibrium bidding strategy of an arbitrary bidder i. A critical notion in Milgrom and Weber (1982)[31] is the first order statistic Y_1, which is the random variable denoting the highest signal received by bidder i's opponents. The following definition generalizes the first-order statistic notion to position auctions:

[12] It will be shown that symmetry is a necessary condition for any equilibrium to be efficient in both one-dimensional assortative position auctions and K-dimensional assortative position auctions (Lemma 1 and Lemma 4), so restricting attention to symmetric equilibria does not lose generality in the efficiency analysis.

Definition 1. *For any arbitrary bidder i, let X be the random variable representing bidder i's own signal x_i. For all $k \in \{1, 2, \cdots, K\}$, let Y_k be the k-th order statistic representing the k-th highest signal received by bidder i's opponents. Let $G_k(y_k|x_i)$ be the conditional distribution of statistic Y_k given $X = x_i$, and let $g_k(y_k|x_i)$ be the associated density function. Let $v^k(x_i, y_k)$ be bidder i's expected value of a click conditional on bidder i's signal x_i and the k-th order statistic that takes value y_k:*

$$v^k(x_i, y_k) = E\big[v(x_i, x_{-i})\big|X = x_i, Y_k = y_k\big] \tag{4}$$

For every bidder i, the realization of Y_k is the minimum value that the signal of bidder i can take such that bidder i should win a position no lower than the k-th highest position in any efficient allocation. When $Y_k = x_i$, the term $v^k(x_i, x_i)$ represents bidder i's expected value per click conditional on receiving a signal that is just high enough to win position k:

$$v^k(x_i, x_i) = E\big[v(x_i, x_{-i})\big|X = x_i, Y_k = x_i\big] \tag{5}$$

It is straightforward to see that for any $k, j \in \{1, 2, \cdots, K\}$, if $k < j$, then $v^k(x_i, x_i) \leq v^j(x_i, x_i)$, given any x_i. The inequality is strict under non-trivially interdependent values. That is, for any bidder i, given any signal x_i, the expected value per click conditional on just being able to win a higher ranked position is lower than that conditional on just being able to win a lower ranked position. This generalizes the "winner's curse" concept in Milgrom and Weber (1982) in the following sense: at any monotonic equilibrium, winning a higher ranked position conveys worse information than winning a lower ranked position.

I next give the definition of ex-post efficient position auction under interdependent values:

Definition 2. *A position auction is ex-post efficient if it always assigns positions in the rank ordering of bidders' ex-post values, given any number of positions K, with any CTR profile $(\alpha_1, \alpha_2, \cdots, \alpha_K)$. Under assumptions **A1-A3**, a position auction is* *ex-post efficient if it always assigns positions in the rank ordering of bidder's private signals.*

In the next section, I will explore how the introduction of interdependent values and the presence of the "winner's curse" affect efficiency of the two existing position auctions that are widely used in practice - GSP auctions and VCG-like auctions.

4 Inefficiency of One-dimensional Position Auctions

A unique feature of position auctions is that each bidder's value from getting a click does not depend on the position of her advertisement[13]. Based on this assumption, the commonly-used GSP auction adopts a simple bidding language that only requires each bidder to submit a one-dimensional bid based on her value per click from any position and computes her bid profile by scaling her bid by the click-through rates of the K positions, instead of asking each bidder to bid for each position separately.

In this section, I show that both GSP auctions and VCG-like auctions with one-dimensional bidding language can be inefficient when there are at least two positions under certain CTR profiles. I begin the analysis by characterizing the allocation rule and payment rule in GSP auctions and VCG-like auctions.

4.1 One-dimensional Position Auctions

A position auction $(\tilde{\mu}, \tilde{p})$ that adopts one-dimensional bids $(b_1, b_2, \cdots, b_N) \in \mathbb{R}^N$, in which $b_i \in \mathbb{R}$ represents bidder i's bid per click for any position, is called a *one-dimensional position auction.* The allocation rule $\tilde{\mu}_i(b_1, b_2, \cdots, b_N) =$ $\left(\tilde{\mu}_i^{(1)}(b_1, b_2, \cdots, b_N), \tilde{\mu}_i^{(2)}(b_1, b_2, \cdots, b_N), \cdots, \tilde{\mu}_i^{(K)}(b_1, b_2, \cdots, b_N)\right)$ is a vector of K functions, in which $\tilde{\mu}_i^{(k)}(b_1, b_2, \cdots, b_N) : \mathbb{R}^{\mathbb{N}} \to [0, 1]$ maps a profile of bids $(b_1, b_2, \cdots, b_N)$

[13]Goldman and Rao (2014)[20] use experimental data to test this assumption and get supportive result.

to the probability that bidder i wins position k. The payment rule $\tilde{p}_i(b_1, b_2, \cdots, b_N) = \left(\tilde{p}_i^{(1)}(b_1, b_2, \cdots, b_N), \tilde{p}_i^{(2)}(b_1, b_2, \cdots, b_N), \cdots, \tilde{p}_i^{(K)}(b_1, b_2, \cdots, b_N)\right)$ is a vector of K functions, in which $\tilde{p}_i^{(k)}(b_1, b_2, \cdots, b_N) : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ maps a profile of bids to the payment of bidder i for position k.

For an arbitrary bidder i, given her opponents' bids b_{-i}, define $\hat{b}^k(b_{-i})$ as the k-th highest bid in b_{-i}, which implies $\hat{b}^1(b_{-i}) \geq \hat{b}^2(b_{-i}) \geq \cdots \geq \hat{b}^K(b_{-i})$. For any $k \geq 1$, if there are $n \geq 2$ equivalent k-th highest bids in b_{-i}, then $\hat{b}^k(b_{-i})$, $\hat{b}^{k+1}(b_{-i})$, ..., $\hat{b}^{k+n-1}(b_{-i})$ are assigned randomly for those n equivalent bids. A one-dimensional position auction is *assortative* if it assigns the k-th highest position to the bidder who submits the k-th highest bid.

Definition 3. *In a one-dimensional position auction $(\tilde{\mu}, \tilde{p})$, the allocation rule $\tilde{\mu}$ is assortative if for all $k \in \{1, 2, \cdots, K\}$,*

$$\tilde{\mu}_i^{(k)}(b_i, b_{-i}) = \begin{cases} 1 & \text{if} \quad \hat{b}^k(b_{-i}) \leq b_i < \hat{b}^{k-1}(b_{-i}) \\ 0 & \text{else} \end{cases} \tag{6}$$

Any tie is broken randomly.

Definition 4. *The one-dimensional GSP auction is characterized by the one-dimensional assortative allocation rule and the GSP payment rule given below. For all $k \in \{1, 2, \cdots, K\}$,*

$$\tilde{p}_i^{G,(k)}(b_i, b_{-i}) = \begin{cases} \alpha_k \hat{b}^k(b_{-i}) & \text{if} \quad \hat{b}^k(b_{-i}) \leq b_i < \hat{b}^{k-1}(b_{-i}) \\ 0 & \text{else} \end{cases} \tag{7}$$

Next, I define a VCG-like position auction format called *one-dimensional VCG auction* that is analogous to the second-price auction under the context of interdependent values single-unit auction in Milgrom and Weber (1982)[31].

Definition 5. *The one-dimensional VCG auction is characterized by the one-dimensional assortative allocation rule and a VCG-like payment rule given below. For all $k \in$*

$\{1, 2, \cdots, K\}$,

$$\tilde{p}_i^{V,(k)}(b_i, b_{-i}) = \begin{cases} \sum_{j=k}^{K}(\alpha_j - \alpha_{j+1})\hat{b}^j(b_{-i}) & \textit{if} \quad \hat{b}^k(b_{-i}) \leq b_i < \hat{b}^{k-1}(b_{-i}) \\ 0 & \textit{else} \end{cases} \tag{8}$$

Although the single-unit second-price auction analyzed by Milgrom and Weber (1982)[31] admits a Bayesian equilibrium that always implements efficient allocation under assumptions **A1**-**A3**, I will show that an analogous result does not exist for the one-dimensional VCG auction with multiple positions and non-trivially interdependent values.

4.2 Characterization of Equilibrium

I start the efficiency analysis by providing a necessary and sufficient condition for existence of an efficient Bayesian equilibrium in any one-dimensional assortative position auction.

Lemma 1. *A one-dimensional position auction $(\tilde{\mu}, \tilde{p})$ with assortative allocation rule is efficient if and only if there exists a symmetric equilibrium in which each bidder's bidding strategy $\beta(x_i)$ is strictly increasing in x_i, for any number of positions K, with any CTR profile $(\alpha_1, \alpha_2, \cdots, \alpha_K)$.*

Proof. See Appendix. □

I will next develop the main result of this section: both the one-dimensional GSP auction and the one-dimensional VCG auction can be inefficient when bidders have interdependent values. Note that it is sufficient to show inefficiency can occur with $K = 2$ positions. For both of the one-dimensional GSP auction (G) and the one-dimensional VCG auction (V), I first provide a necessary condition for any monotonic bidding strategy $\beta^L(x_i)$ to be a Bayesian equilibrium of the auction $L \in \{G, V\}$ with two positions, and then finish the analysis by showing that the unique $\beta^L(x_i)$ characterized by this equilibrium condition cannot be monotonic under some CTR profiles.

Lemma 2. *In the one-dimensional GSP auction with two positions, if an efficient Bayesian equilibrium bidding strategy* $\beta^G(x_i)$ *exists, then* $\beta^G(x_i)$ *is characterized as below:*

For all $x_i \in [0, \bar{x}]$, $\beta^G(x_i)$ *satisfies the Volterra equation*

$$\beta^G(x_i) = \frac{g_1(x_i|x_i)\Big[(\alpha_1 - \alpha_2)v^1(x_i, x_i) + \alpha_2 \int_0^{x_i} \beta^G(y_2) g_{2|1}(y_2|x_i, x_i) dy_2\Big] + g_2(x_i|x_i)\alpha_2 v^2(x_i, x_i)}{\alpha_1 g_1(x_i|x_i) + \alpha_2 g_2(x_i|x_i)} \tag{9}$$

Proof. See Appendix. □

Lemma 3. *In the one-dimensional VCG auction with two positions, if an efficient Bayesian equilibrium bidding strategy* $\beta^V(x_i)$ *exists, then* $\beta^V(x_i)$ *is characterized as below:*

For all $x_i \in [0, \bar{x}]$,

$$\beta^V(x_i) = \frac{g_1(x_i|x_i)(\alpha_1 - \alpha_2)v^1(x_i, x_i) + g_2(x_i|x_i)\alpha_2 v^2(x_i, x_i)}{g_1(x_i|x_i)(\alpha_1 - \alpha_2) + g_2(x_i|x_i)\alpha_2} \tag{10}$$

Proof. See Appendix. □

To better understand the characterization of equilibria in Lemma 2 and Lemma 3, let $\Pi_1^L(x_i, y_1, y_2)$ and $\Pi_2^L(x_i, y_1, y_2)$ denote the expected payoffs from winning position 1 and 2 in auction $L \in \{G, V\}$ respectively, given the realizations of $X = x_i, Y_1 = y_1, Y_2 = y_2$. The equilibrium bidding strategy $\beta^G(x_i)$ characterized in Lemma 2 is derived from the following equilibrium condition:

$$
\begin{aligned}
&g_1(x_i|x_i)\underbrace{\left[(\alpha_1-\alpha_2)v^1(x_i,x_i)-\alpha_1\beta^G(x_i)+\alpha_2\int_0^{x_i}\beta^G(y_2)g_{2|1}(y_2|x_i,x_i)dy_2\right]}_{E[\Pi_1^G-\Pi_2^G|X=x_i,Y_1=x_i]}\\
&+g_2(x_i|x_i)\underbrace{\left[\alpha_2 v^2(x_i,x_i)-\alpha_2\beta^G(x_i)\right]}_{E[\Pi_2^G|X=x_i,Y_2=x_i]}=0
\end{aligned}
\tag{11}
$$

Similarly, the equilibrium bidding strategy $\beta^V(x_i)$ characterized in Lemma 3 is derived from the following equilibrium condition:

$$
g_1(x_i|x_i)\underbrace{\left[(\alpha_1-\alpha_2)\big(v^1(x_i,x_i)-\beta^V(x_i)\big)\right]}_{E[\Pi_1^V-\Pi_2^V|X=x_i,Y_1=x_i]}+g_2(x_i|x_i)\underbrace{\left[\alpha_2\big(v^2(x_i,x_i)-\beta^V(x_i)\big)\right]}_{E[\Pi_2^V|X=x_i,Y_2=x_i]}=0
\tag{12}
$$

Note that in the special case of independent private values where $v^k(x_i,x_i)=x_i$ for all k, the equilibrium of the one-dimensional VCG auction is given by $\beta^V(x_i)=x_i$, consistent with the dominant strategy equilibrium in the VCG auction under independent private values. In the special case of $\alpha_2=0$, the equilibrium $\beta^V(x_i)=v^1(x_i,x_i)$ is consistent with the symmetric equilibrium of the second-price auction in Milgrom and Weber (1982)[31]. In the special case of $\alpha_1=\alpha_2$, the equilibrium $\beta^V(x_i)=v^2(x_i,x_i)$ is consistent with the equilibrium of the uniform-price auction with single-unit demands.

Equations (11) and (12) imply that in both one-dimensional GSP auctions and one-dimensional VCG auctions with two positions, for an arbitrary bidder i with signal x_i, the net impact of winning position 1 instead of position 2 on the margin of $Y_1=x_i$ and winning position 2 instead of nothing on the margin of $Y_2=x_i$ weighted by corresponding probability masses must equal zero at any efficient equilibrium. For all $x_i\in[0,\bar{x}]$,

$$
g_1(x_i|x_i)E\left[\Pi_1^L-\Pi_2^L\middle|X=x_i,Y_1=x_i\right]+g_2(x_i|x_i)E\left[\Pi_2^L\middle|X=x_i,Y_2=x_i\right]=0,\quad L\in\{G,V\}
\tag{13}
$$

The intuition behind this equilibrium condition is that in a one-dimensional assortative position auction, for any bidder i, increasing bid increases the probability of winning position 1 instead of position 2 and the probability of winning position 2 instead of nothing at the same time, so each bidder's optimal strategy $\beta^L(x_i)$ must balance the trade-offs between every pair of adjacent positions at corresponding margins. I next show that the unique $\beta^L(x_i)$ satisfying this equilibrium condition cannot be monotonic under some CTR profile, for both $L = G, V$.

4.3 Efficiency Analysis

The following two propositions present the main result of section 4:

Proposition 1. *For any value function $v_i(x_i, x_{-i})$ satisfying assumptions **A1-A3**, there exists some number of positions K with some CTR profile under which no efficient Bayesian equilibrium exists in the one-dimensional GSP auction.*

Proof. See Appendix. □

Proposition 2. *For any non-trivially interdependent value function $v_i(x_i, x_{-i})$ satisfying assumptions **A1-A3** and $\frac{\partial v_i(x_i,x_{-i})}{\partial x_j} \neq 0$ for $j \neq i$, there exists some number of positions K with some CTR profile under which no efficient Bayesian equilibrium exists in the one-dimensional VCG auction.*

Proof. See Appendix. □

The intuition behind Proposition 1 and Proposition 2 is that in both one-dimensional GSP auctions and one-dimensional VCG auctions with two positions, there exists some CTR profile under which the superior position is less desirable than the inferior position given expected payoffs, which leads to differential bid-shading incentives across bidders and results in non-existence of monotonic equilibrium bidding strategy. The following analysis elaborates this intuition in each auction.

The source of inefficiency of one-dimensional GSP auctions comes from both its simple payment rule and the presence of the generalized "winner's curse" under interdependent values. In a one-dimensional GSP auction with two positions, when the click rate of the second position is close to that of the first position, each bidder's expected payoff conditional on winning the first position is lower than that conditional on winning the second position for two reasons. First, a bidder receives similar number of clicks from winning either position but pays a much higher price for each click if she wins the first position given the GSP payment rule. Second, the expected value for every click conditional on winning a higher ranked position is lower than that conditional on winning a lower ranked position. Therefore, at any monotonic equilibrium, the second position is more desirable than the first position when α_2 is sufficiently close to α_1. Under the one-dimensional bidding language, each bidder is forced to submit the same bid for both positions, so the equilibrium bid must balance net trade-offs between all pairs of adjacent positions weighted by corresponding probability masses $g_k(x_i|x_i)$ that varies with signal x_i, as shown in equation (13). Because the weight attached to $E[\Pi_1^G - \Pi_2^G | X = x_i, Y_1 = x_i]$ is higher for bidders with higher signals x_i compared to those with lower signals, the bid-shading incentive is stronger for the former. This differentiated bid-shading incentive across bidders' signals can lead to violation of monotonicity in the unique equilibrium $\beta^G(x_i)$ characterized in Lemma 2. Therefore, a symmetric and strictly increasing equilibrium bidding strategy does not exist under certain CTR profiles, which implies the inefficiency of the GSP auction.

Proposition 1 not only confirms the inefficiency of the GSP auction under private values as pointed out by Gomes and Sweeney (2014)[21], but also identifies an additional source of inefficiency under the broader information structure of interdependent values. The non-existence of monotonic equilibrium not only comes from the oversimplified GSP payment rule, but also comes from the oversimplified single-dimensional bidding language. Proposition 2 provides further support for this argument by showing that with the one-dimensional bidding language, simply modifying the GSP payment rule to the more complicated VCG-like payment rule does not eliminate inefficiency under interdependent values, as differentiated bid-shading incentives across bidders still exist in the one-dimensional VCG auction.

The source of inefficiency of one-dimensional VCG auctions arises from the generalized "winner's curse" under interdependent values. Similar to the one-dimensional GSP auction, the weight attached to trade-offs between each pair of adjacent positions $g_k(x_i|x_i)$ varies in x_i in the one-dimensional VCG auction. Under the VCG-like payment rule, it is optimal for each bidder to bid her true expected value per click conditional on $Y_2 = x_i$ if the probability of winning the first position is zero so that only the trade-off between winning the second position and nothing needs to be considered. However, for any bidder who receives a signal $x_i > 0$, there is positive probability of winning the first position at any monotonic equilibrium. With non-trivially interdependent values, the expected value $v^k(x_i, x_i)$ differs across positions, with $v^1(x_i, x_i) < v^2(x_i, x_i)$ under the generalized "winner's curse." Therefore, every bidder with $x_i > 0$ shades bid below $v^2(x_i, x_i)$. Bidders with higher signals have stronger bid-shading incentive, since they need to weigh the impact of the generalized "winner's curse" more significantly given that they are more likely to win the first position when other bidders bid monotonically. This differentiated bid-shading incentive can lead to non-monotonicity of the unique equilibrium bidding strategy $\beta^V(x_i)$ characterized in Lemma 3, which implies that the one-dimensional VCG auction can also be inefficient under some CTR profile. Moreover, the non-existence of monotonic equilibrium in the one-dimensional VCG auction also tend to occur when α_2 is close to α_1, as the bid-shading incentive under the generalized "winner's curse" is amplified when the quality of the superior position is not significantly better than the quality of the inferior position. In addition to inefficiency, another problem with the one-dimensional GSP and VCG auctions is that the unique equilibrium bidding strategy is very complicated under interdependent values, which may further decrease efficiency when bidders are not sophisticated enough and deviate from the equilibrium strategies by mistake.

To summarize this section, it can be concluded that a common source of inefficiency of the one-dimensional GSP auction and the one-dimensional VCG auction comes from the fact that bidders have bid-shading incentives to avoid winning the highest ranked position, while both auctions use a simple one-dimensional bidding language that restrict bidders to submit the same bid for all positions. This restriction requires each bidder's equilibrium bid to balance the net trade-offs between all pairs of adjacent positions on different margins, which is impossible for any monotonic bidding strategy under certain CTR profiles. It is natural to conjecture that allowing bidders to submit different bids for each position such that the equilibrium bid for each position k balances only the trade-off between position k and position $k+1$ conditional on $Y_k = x_i$ may resolve the inefficiency problem. The next section confirms this conjecture.

5 Efficiency of K-dimensional Position Auctions

In this section, I propose a modification of the one-dimensional GSP auction and the one-dimensional VCG auction by allowing each bidder to submit a vector of K-dimensional bid for K positions. I show that both modified auctions have unique efficient Bayesian equilibria given any number of positions, with any CTR profile. Moreover, the Generalized English Auction that implicitly adopts a K-dimensional bidding language has a unique efficient ex-post equilibrium.

5.1 K-dimensional Position Auctions

I first construct a class of position auctions that adopts a K-dimensional bidding language and a K-dimensional assortative allocation rule that corresponds to the assortative allocation rule in one-dimensional position auctions. A position auction (μ, p) that adopts K-dimensional bids $(b_1, b_2, \cdots, b_N) \in \mathbb{R}^K \times \mathbb{R}^N$, in which $b_i \in \mathbb{R}^K$ represents bidder i's per-click bid for every position $k \in \{1, 2, \cdots, K\}$, is called a *K-dimensional position auction*. The allocation rule $\mu_i(b_1, b_2, \cdots, b_N) = \Big(\mu_i^{(1)}(b_1, b_2, \cdots, b_N), \mu_i^{(2)}(b_1, b_2, \cdots, b_N), \cdots, \mu_i^{(K)}(b_1, b_2, \cdots, b_N)\Big)$ is a vector of K functions, in which $\mu_i^{(k)}(b_1, b_2, \cdots, b_N) : \mathbb{R}^K \times \mathbb{R}^N \to [0, 1]$ maps a profile of bids $(b_1, b_2, \cdots, b_N)$

to the probability that bidder i wins position k. The payment rule $p_i(b_1, b_2, \cdots, b_N) = \Big(p_i^{(1)}(b_1, b_2, \cdots, b_N), p_i^{(2)}(b_1, b_2, \cdots, b_N), \cdots, p_i^{(K)}(b_1, b_2, \cdots, b_N)\Big)$ is a vector of K functions, in which $p_i^{(k)}(b_1, b_2, \cdots, b_N) : \mathbb{R}^K \times \mathbb{R}^N \to \mathbb{R}$ maps a profile of bids to the payment of bidder i for position k.

For any position k, define $S_k(b_1, b_2, \cdots, b_N)$ as the set of bidders who should win some position strictly above the k-th highest position at bidding profile $(b_1, b_2, \cdots, b_N)$ according to the allocation rule of the auction:

$$S_k(b_1, b_2, \cdots, b_N) = \Big\{j \in \{1, 2, \cdots, N\} \,\Big|\, \exists\ k' < k \text{ s.t. } \mu_j^{(k')}(b_j, b_{-j}) = 1\Big\} \tag{14}$$

For any arbitrary bidder i, given any profile of K-dimensional bids (b_i, b_{-i}), define $\max\big\{b^k_{-i/S_k(b_i,b_{-i})}\big\}$ as the highest bid for position k among bidder i's opponents who do not win any position above k. A K-dimensional position auction is *assortative* if its allocation rule is characterized by the following definition:

Definition 6. *In a K-dimensional position auction* (μ, p)*, the allocation rule* μ *is assortative if for all* $k \in \{1, 2, \cdots, K\}$*,*

$$\mu_i^{(k)}(b_i, b_{-i}) = \begin{cases} 1 & \textit{if} \quad i \notin S_k,\ \max\big\{b^k_{-i/S_k(b_i,b_{-i})}\big\} \leq b_i^k \\ 0 & \textit{else} \end{cases} \tag{15}$$

Any tie is broken randomly.

In an assortative K-dimensional position auction (μ, p), each bidder submits a vector of K bids $(b_i^1, b_i^2, \cdots, b_i^K)$ simultaneously in a sealed-bid format. The auctioneer collects all bids at once and assigns the first position to the bidder who submits the highest bid for position 1, the second position to the bidder who submits the highest bid for position 2, among those who do not win position 1, etc. Once a bidder is assigned a position k, her bids for lower positions b_i^j with $j > k$ will not be considered in the allocation of lower positions.

I next construct two assortative K-dimensional position auctions that can be viewed as modified one-dimensional GSP and VCG auction, respectively. I call these auctions K-dimensional GSP auction and K-dimensional VCG auction.

Definition 7. *The K-dimensional GSP auction is characterized by the assortative K-dimensional allocation rule and the payment rule given below. For all* $k \in \{1, 2, \cdots, K\}$*,*

$$p_i^{G,(k)}(b_i, b_{-i}) = \begin{cases} \alpha_k \max\left\{b^k_{-i/S_k(b_i,b_{-i})}\right\} & \text{if} \quad i \notin S_k,\ \max\left\{b^k_{-i/S_k(b_i,b_{-i})}\right\} \leq b_i^k \\ 0 & \text{else} \end{cases} \tag{16}$$

Definition 8. *The K-dimensional VCG auction can be characterized by the assortative K-dimensional allocation rule and the payment rule given below. For all* $k \in \{1, 2, \cdots, K\}$*,*

$$p_i^{V,(k)}(b_i, b_{-i}) = \begin{cases} \sum_{j=k}^{K}(\alpha_j - \alpha_{j+1}) \max\left\{b^j_{-i/S_j(b_i,b_{-i})}\right\} & \text{if} \quad i \notin S_k,\ \max\left\{b^k_{-i/S_k(b_i,b_{-i})}\right\} \leq b_i^k \\ 0 & \text{else} \end{cases} \tag{17}$$

In addition to the two K-dimensional position auctions proposed above, there exists another auction that also effectively uses a K-dimensional bidding language - the Generalized English Auction characterized by Edelman et al. (2007)[15] also implicitly allows bidders to condition their bids on positions. In a Generalized English Auction, there is a continuously ascending clock showing the current price. Initially, all advertisers are in the auction. An advertiser can drop out at any time, and her bid is the price on the clock when she drops out. The auction ends when there is only one bidder left. This last bidder wins the first position, and her per click payment equals to the next-to-last bidder's drop-out price. The next-to-last bidder wins the second position, and her per click payment equals to the third highest bid, etc. Any tie is broken randomly when bidders drop out simultaneously. Since all drop-out prices are observable, each bidder can update bid strategy every time an opposing bidder drops out, which implicitly allows for a K-dimensional bidding language.

5.2 Characterization of Equilibria and Efficiency Analysis

To begin the efficiency analysis of K-dimensional position auctions, I first provide a necessary and sufficient condition for any K-dimensional assortative position auction to be efficient:

Lemma 4. *A K-dimensional position auction* (μ, p) *with assortative allocation rule is efficient if and only if given any number of positions* K*, there exists a symmetric equilibrium in which each bidder's bidding strategy* $\big(\beta_1(x_i), \beta_2(x_i), \cdots, \beta_K(x_i)\big)$ *satisfies* $\beta_k'(x_i) > 0$ *for every position* $k \in \{1, 2, \cdots, K\}$*, under any CTR profile* $(\alpha_1, \alpha_2, \cdots, \alpha_K)$*.*

Proof. See Appendix. □

Next, I develop the main result of this section: the K-dimensional GSP auction, the K-dimensional VCG auction, and the Generalized English Auction are always efficient given any value function satisfying assumptions **A1**-**A3**, for any number of positions K, with any CTR profile. I first characterize the unique symmetric equilibria of the K-dimensional GSP auction, the K-dimensional VCG auction and the Generalized English Auction. It will be shown that the equilibria of all three auctions satisfy the necessary and sufficient condition in Lemma 4.

5.2.1 Equilibrium of K-dimensional GSP Auction

The unique symmetric Bayesian equilibrium bidding strategy in the K-dimensional GSP auction is given in Proposition 3:

Proposition 3. *Define the K-dimensional bidding strategy* $\beta(x_i) = \big(\beta_1(x_i), \beta_2(x_i), \cdots, \beta_K(x_i)\big)$ *as follows:*

$$\beta_K(x_i) = v^K(x_i, x_i) \tag{18}$$

for the last position K.

$$\beta_k(x_i) = v^k(x_i, x_i) - \frac{\alpha_{k+1}}{\alpha_k}\Big[v^k(x_i, x_i) - \int_0^{x_i} \beta_{k+1}(y_{k+1}) dG_{k+1}\big(y_{k+1}\big|X = x_i, Y_k = x_i\big)\Big] \tag{19}$$

for any position $k \in \{1, 2, \cdots, K-1\}$.

Let $b_i^* = \beta(x_i) = \big(\beta_1(x_i), \beta_2(x_i), \cdots, \beta_K(x_i)\big)$ for each bidder i, then the n-tuple of strategies $(b_1^*, b_2^*, \cdots, b_N^*)$ is a Bayesian Nash equilibrium of the K-dimensional GSP auction.

Proof. See Appendix. □

Proposition 3 shows that the equilibrium bid for the last position K in the K-dimensional GSP auction is the expected value per click conditional on receiving a signal just high enough to win the last position, $Y_k = x_i$. On the other hand, the equilibrium bid for any position above the last position in the K-dimensional GSP auction is the expected value per click subtracted by the expected payoff from winning the next position divided by α_k, conditional on $Y_k = x_i$. The subtracted term can be interpreted as the per-click opportunity cost of winning position k. Since $\beta_K(x_i) = v^K(x_i, x_i)$ is strictly increasing in x_i, and

$$\frac{d\beta_k(x_i)}{dx_i} = \Big(1 - \frac{\alpha_{k+1}}{\alpha_k}\Big)\frac{\partial v^k(x_i, x_i)}{\partial x_i} + \frac{\alpha_{k+1}}{\alpha_k}\beta_{k+1}(x_i) g_{k+1}(x_i|x_i, x_i) > 0, \quad \forall k \in \{1, 2, \cdots, K-1\} \tag{20}$$

The symmetric equilibrium bidding strategy $\beta_k(x_i)$ for every position k is strictly increasing in x_i. According to Lemma 4, the K-dimensional GSP auction is always efficient.

Corollary 1. *The K-dimensional GSP auction always implements the ex-post efficient allocation in a symmetric Bayesian equilibrium given any value function $v_i(x_i, x_{-i})$ satisfying assumptions **A1**-**A3**, for any number of positions K, with any CTR profile $(\alpha_1, \alpha_2, \cdots, \alpha_K)$.*

To better understand the equilibrium characterized in Proposition 3, let $\Pi_k^G(x_i, y_1, \cdots, y_{N-1})$ denote the payoff of winning position k given realizations $X = x_i, Y_1 = y_1, \cdots, Y_{N-1} = y_{N-1}$ in the K-dimensional GSP auction. The equilibrium bidding strategy $\big(\beta_1(x_i), \beta_2(x_i), \cdots, \beta_K(x_i)\big)$ solves

$$\underbrace{\alpha_k\Big[v^k(x_i, x_i) - \beta_k(x_i)\Big]}_{E[\Pi_k^G|X=x_i,Y_k=x_i]} = \underbrace{\alpha_{k+1}\Big[v^k(x_i, x_i) - \int_0^{x_i} \beta_{k+1}(y_{k+1}) dG_{k+1|k}\big(y_{k+1}|x_i, x_i\big)\Big]}_{E[\Pi_{k+1}^G|X=x_i,Y_k=x_i]}, \quad \forall k \in \{1, 2, \cdots, K\} \tag{21}$$

which implies that at the symmetric equilibrium of the K-dimensional GSP auction, each bidder should be indifferent between winning position k and position $k+1$ conditional on $Y_k = x_i$, at which value her signal is just high enough to win position k, for any position $k \in \{1, 2, \cdots, K\}$.

5.2.2 Equilibrium of K-dimensional VCG Auction

The unique symmetric Bayesian equilibrium bidding strategy in the K-dimensional VCG auction is given by Proposition 4:

Proposition 4. *Let $\beta_k(x_i) = v^k(x_i, x_i)$ for all $k \in \{1, 2, \cdots, K\}$. Let $b_i^* = \beta(x_i) = (\beta_1(x_i), \beta_2(x_i), \cdots, \beta_K(x_i))$, then the n-tuple of strategies $(b_1^*, b_2^*, \cdots, b_N^*)$ is a Bayesian Nash equilibrium of the K-dimensional VCG auction.*

Proof. See Appendix. □

Since $v^k(x_i, x_i)$ is strictly increasing in x_i for all $k \in \{1, 2, \cdots, K\}$, the K-dimensional VCG auction is always efficient.

Corollary 2. *The K-dimensional VCG auction always implements the ex-post efficient allocation in a symmetric Bayesian equilibrium given any value function $v_i(x_i, x_{-i})$ satisfying assumptions **A1-A3**, for any number of positions K, with any CTR profile $(\alpha_1, \alpha_2, \cdots, \alpha_K)$.*

To better understand the equilibrium bidding strategy characterized in Proposition 4, let $\Pi_k^V(x_i, y_1, \cdots, y_{N-1})$ denote the payoff of winning position k given realizations $X = x_i, Y_1 = y_1, \cdots, Y_{N-1} = y_{N-1}$ in the K-dimensional VCG auction. The equilibrium bidding strategy $\big(\beta_1(x_i), \beta_2(x_i), \cdots, \beta_K(x_i)\big)$ in the K-dimensional VCG auction solves

$$\underbrace{(\alpha_k - \alpha_{k+1})\Big[v^k(x_i, x_i) - \beta_k(x_i)\Big]}_{E[\Pi_k^V - \Pi_{k+1}^V | X = x_i, Y_k = x_i]} = 0, \quad \forall k \in \{1, 2, \cdots, K\} \tag{22}$$

which implies that at the equilibrium of K-dimensional VCG auction, each bidder with signal x_i is indifferent between winning position k and position $k+1$ when $Y_k = x_i$, for all position k. Comparing equation (21) and equation (22), it follows that the equilibria of K-dimensional GSP auction and K-dimensional VCG auction can be characterized by the same condition:

$$E\Big[\Pi_k^L - \Pi_{k+1}^L \Big| X = x_i, Y_k = x_i\Big] = 0, \quad \forall\, k \in \{1, 2, \cdots, K\}, \quad \forall\, L \in \{G, V\} \tag{23}$$

Equation (23) shows that with K-dimensional bidding language, each bidder submits K separate bids such that the bid for position k balances only the trade-off between position k and position $k+1$ conditional on $Y_k = x_i$, in contrast to the equilibrium condition in one-dimensional position auctions characterized by equation (13). The differentiated bid-shading incentive across bidders' signals in the one-dimensional auctions is replaced by the differentiated bid-shading incentive across positions in the K-dimensional auctions, which resolves the inefficiency problem.

Comparing the K-dimensional position auctions to their one-dimensional counterparts also demonstrates that adopting a K-dimensional bidding language simplifies each bidder's equilibrium bidding strategy. The equilibrium bidding strategy of the K-dimensional auctions characterized in Proposition 3 and Proposition 4 are easier to compute compared to the equilibrium of the one-dimensional counterparts characterized in Lemma 2 and Lemma 3, which implies that adopting a more complicated auction design can actually simplifies bidders' equilibrium bidding strategy. The trade-off between simplicity in auction design and simplicity in equilibrium bidding strategy can also be shown by comparing the equilibrium of the K-dimensional GSP auction and K-dimensional VCG auction: the K-dimensional GSP auction has a simpler payment rule but a more complicated equilibrium bidding strategy compared to the K-dimensional VCG auction.

The next example provides an illustration of the Bayesian equilibrium bidding strategies in the K-dimensional VCG auction and K-dimensional GSP auction.

Example 1. *Consider the K-dimensional VCG auction and K-dimensional GSP auction with* $K = 2$ *positions and* $N = 3$ *bidders, with CTR profile normalized to* $(1, \alpha_2)$*. The bidders' private signals are independently and identically drawn from the uniform distribution on* $[0, 1]$*. Bidder i's value per click* v_i *is a function of her own signal* x_i *and her opponents' signals* x_j, x_k*:*

$$v_i(x_i, x_j, x_k) = \lambda x_i + \frac{1-\lambda}{2}\big(x_j + x_k\big), \quad \lambda \in \left[\frac{1}{3}, 1\right] \tag{24}$$

Figure 1 plots the equilibrium bidding strategy $\big(\beta_1^V(x), \beta_2^V(x)\big)$ *in the K-dimensional VCG auction and* $\big(\beta_1^G(x), \beta_2^G(x)\big)$ *in the K-dimensional GSP auction, under different values of* $\lambda \in \{1, \frac{1}{2}, \frac{1}{3}\}$ *and* $\alpha_2 \in \{0.75, 0.25\}$*.*

Figure 1 provides two main insights. First, comparing the equilibria under values of $\lambda = 1, \frac{1}{2}, \frac{1}{3}$ given the same α_2 illustrates the impact of increasing degree of interdependency among bidders' values on the equilibria of the two auctions. Since $\beta_1^L(x_i) \leq \beta_2^L(x_i)$ for both auctions $L \in \{G, V\}$ under any α_2, the equilibrium bid of any bidder for position 1 is weakly lower than that for position 2 in both auctions. The difference $\Big(\beta_2^L(x_i) - \beta_1^L(x_i)\Big)$ is increasing in x_i in the K-dimensional GSP auction, while stays constant in x_i in the K-dimensional VCG auction. Moreover, $\Big(\beta_2^L(x_i) - \beta_1^L(x_i)\Big)$ is greater in both auctions when λ is lower, which means the degree of bid-shading for position 1 is more significant in both auctions when the degree of interdependency in values is stronger and the impact of the generalized "winner's curse" is more significant.

Second, comparing the equilibria under $\alpha_2 = 0.75$ to $\alpha_2 = 0.25$ under the same value of λ shows the impact of increasing difference in CTR between the superior

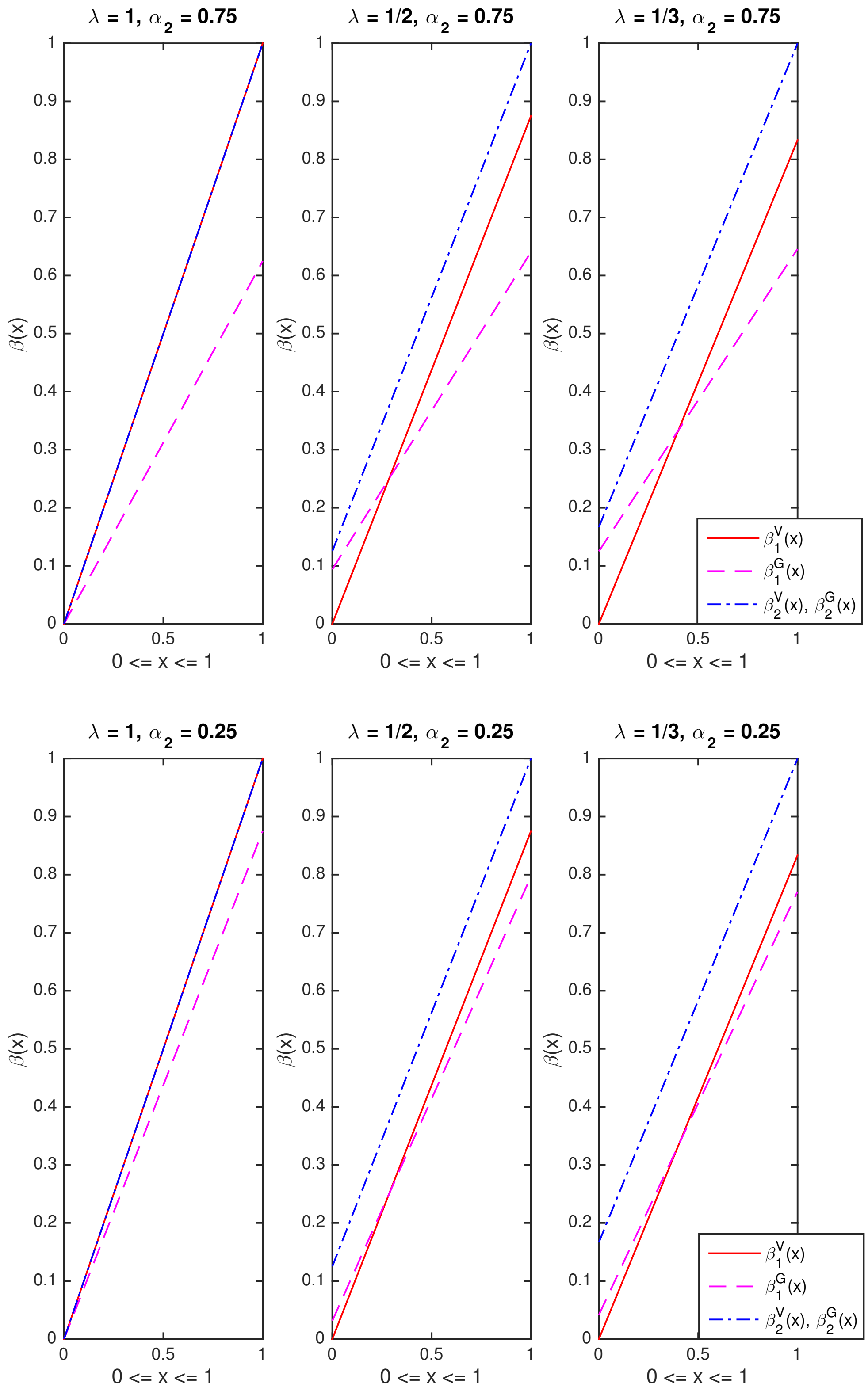

Figure 1: Equilibrium Bidding Strategies in K-dimensional VCG and GSP Auction

position and the inferior position on the equilibria of the two auctions. It can be shown that under the same λ, $\left(\beta_2^G(x_i) - \beta_1^G(x_i)\right)$ increases in α_2 as well as in x_i in the K-dimensional GSP auction, while $\left(\beta_2^V(x_i) - \beta_1^V(x_i)\right)$ remains unaffected by α_2 and stays constant in x_i in the K-dimensional VCG auction. Therefore, the bid-shading incentive for position 1 is greater when the CTR of two positions are closer in the K-dimensional GSP auction, while the equilibrium bids are unaffected by CTR in the K-dimensional VCG auction.

5.2.3 Equilibrium of Generalized English Auction

The next proposition characterizes the unique symmetric equilibrium of the Generalized English Auction (GEA) under interdependent values and shows this dynamic auction with an implicit K-dimensional bidding language is also efficient.

At any time in the auction, let n denote the number of bidders who are still active in the auction, and $(N-n)$ denote the number of bidders who have dropped out. Let $(p_N, p_{N-1} \cdots, p_{n+1})$ denote the drop-out prices of the $(N-n)$ bidders, in which p_N is the bid of the first drop out bidder, and p_{n+1} is the bid of the last drop out bidder at current time, so $p_N \leq p_{N-1} \leq \cdots \leq p_{n+1}$. When there are n remaining bidders in the auction, the equilibrium strategy for bidder i specifies her optimal drop out price given her private signal x_i and a history of drop out prices $(p_N, p_{N-1}, \cdots, p_{n+1})$. Define

$$v^{(k)}(x_i, y_k, y_{k+1}, \cdots, y_{N-1}) = E\left[v_i \middle| X = x_i, Y_k = y_k, Y_{k+1} = y_{k+1}, \cdots, Y_{N-1} = y_{N-1}\right] \tag{25}$$

as bidder i's expected value conditional on her own signal $X = x_i$ and the realization of all of the $(N-k)$ lowest signals among opponents' signals, $Y_k = y_k, Y_{k+1} = y_{k+1}, \cdots, Y_{N-1} = y_{N-1}$. The unique symmetric equilibrium of the GEA under interdependent values is characterized in Proposition 5:

Proposition 5. *Define strategy* $b^* = (b_N^*, b_{N-1}^*, \cdots, b_2^*)$ *as follows:*

$$b_N^*(x_i) = v^{(K)}(x_i, \underbrace{x_i, \cdots, x_i}_{(N-K)})$$

$$b_n^*(x_i|p_N, \cdots, p_{n+1}) =$$

$$\begin{cases} v^{(K)}(x_i, \underbrace{x_i, \cdots, x_i}_{(n-K)}, \underbrace{y_n, y_{n+1}, \cdots, y_{N-1}}_{(N-n)\ lowest\ signals}) & \text{if} \quad (K+1) \le n \le (N-1) \\ v^{(n-1)}(x_i, x_i, \underbrace{y_n, y_{n+1}, \cdots, y_{N-1}}_{(N-n)\ lowest\ signals}) - \frac{\alpha_n}{\alpha_{n-1}}\left[v^{(n-1)}(x_i, x_i, \underbrace{y_n, y_{n+1}, \cdots, y_{N-1}}_{(N-n)\ lowest\ signals}) - p_{n+1}\right] & \text{if} \quad n \le K \end{cases} \tag{26}$$

in which $y_n, y_{n+1}, \cdots, y_{N-1}$ *are calculated from*

$$\begin{aligned} b_N^*(y_{N-1}) &= p_N \\ b_{N-1}^*(y_{N-2}|p_N) &= p_{N-1} \\ &\cdots \\ b_{n+1}^*(y_n|p_N, \cdots, p_{n+2}) &= p_{n+1} \end{aligned} \tag{27}$$

The N*-tupple bidding strategy* $(b^*, ..., b^*)$ *is an ex-post equilibrium of the Generalized English Auction under interdependent values.*

Proof. See Appendix. □

Since the equilibrium bidding strategy $b_n^*(x_i)$ at any stage of the GEA is increasing in x_i, the GEA is also efficient.

Corollary 3. *The Generalized English Auction always implements the ex-post efficient allocation in an ex-post equilibrium, given any value function* $v_i(x_i, x_{-i})$ *satisfying assumptions* ***A1-A3****, for any number of positions* K*, with any CTR profile* $(\alpha_1, \alpha_2, \cdots, \alpha_K)$*.*

To better understand the equilibrium of GEA, let $\Pi_k^E(x_i, y_1, y_2, \cdots, y_{N-1})$ be the payoff from winning position k conditional on $X = x_i, Y_1 = y_1, \cdots, Y_{N-1} = y_{N-1}$. The equilibrium condition of GEA characterized in Proposition 5 can be interpreted as

$$
\begin{aligned}
&E\Big[\Pi_K^E\Big|X = x_i, Y_K = x_i, \cdots, Y_N = x_i\Big] = 0, \text{ if } n = N \\
&E\Big[\Pi_K^E\Big|X = x_i, Y_K = x_i, \cdots, Y_{n-1} = x_i, Y_n = y_n, \cdots, Y_N = y_N\Big] = 0, \text{ if } K+1 \leq n \leq N-1 \\
&E\Big[\Pi_k^E - \Pi_{k+1}^E\Big|X = x_i, Y_k = x_i, Y_{k+1} = y_{k+1}, \cdots, Y_N = y_N\Big] = 0, \text{ if } n = k+1 \leq K
\end{aligned}
\tag{28}
$$

which implies that the optimal drop-out price at any time of the auction must balance the trade-off between winning position k and position $k+1$ conditional on $Y_k = x_i$, given the profile of revealed signals from the history of drop-out prices. When there are more bidders than positions left in the auction, each bidder's optimal drop-out strategy specifies the price at which she is indifferent between winning the lowest position and winning nothing. When there are (weakly) fewer bidders than positions left in the auction, each bidder's optimal drop-out strategy specifies the price at which she is indifferent between winning the next position higher than the current lowest position and winning the current lowest position at the most recent drop-out price. Comparing the equilibrium condition characterized in (28) to the equilibrium condition characterized in (23), it can be shown that the equilibrium condition of GEA is similar to the equilibrium condition of the K-dimensional GSP auction and the K-dimensional VCG auction, while the only difference comes from that each remaining bidder can update her belief from revealed signals of drop-out bidders in GEA. Despite of its efficiency, the GEA cannot be practically implemented in real-time sponsored search market. Nevertheless, by adopting a K-dimensional bidding language, both the GSP auction and the VCG-like auction can implement the same efficient allocation as the GEA under similar equilibrium conditions.

6 Revenue of K-dimensional Position Auctions

In this section, I first provide a revenue ranking of the three efficient K-dimensional position auctions under interdependent values. Then I characterize the optimal position auction as a direct revelation mechanism and compare the expected revenues of the three efficient auctions to the optimal revenue subject to no reserve price.

6.1 Revenue Ranking

The following proposition gives the revenue ranking of the K-dimensional GSP auction, the K-dimensional VCG auction and the GEA.

Proposition 6. *The expected revenue of the Generalized English Auction is higher than the expected revenue of the K-dimensional VCG auction, which in turn equals to the expected revenue of the K-dimensional GSP auction, for any value function $v_i(x_i, x_{-i})$ and distribution of signals $F(x_1, x_2, \cdots, x_N)$ satisfying assumptions **A1**-**A5**.*

$$R^{GEA} \geq R^{K-VCG} = R^{K-GSP} \tag{29}$$

Proof. See Appendix. □

The intuition behind revenue equivalence of the K-dimensional GSP auction and the K-dimensional VCG auction is the following. Both auctions are sealed-bid auctions, so no information is elicited before final allocation and payments are determined. Both auctions adopt the same K-dimensional assortative allocation rule and some variation of a "second-price" payment rule under which each bidder's bid only affect her allocation but not her payment. In the proof of Proposition 6, it is shown that although each bidder's payment in the two auctions depends on her opponents' bids in different ways, bidders are able to incorporate different payment rules into their bidding strategies so that the expected payment for a bidder with the same signal is the same in the two auctions.

The intuition behind the revenue ranking of the GEA and the K-dimensional VCG auction comes from the Linkage Principle in Milgrom and Weber (1982)[31]. With affiliated signals, the dynamic auction performs better than static auctions since part of the signals are elicited during the drop-out process. On the other hand, with independent signals, the GEA is revenue equivalent to the other two static K-dimensional position auctions, which gives the following corollary:

Corollary 4. *When bidders' signals are independently and identically distributed, the Generalized English Auction, the K-dimensional VCG auction and the K-dimensional GSP auction yield the same expected revenue, for any value function $v_i(x_i, x_{-i})$ that satisfies assumptions **A1-A3**.*

Corollary 4 is consistent with the Revenue Equivalence Theorem in auction theory. When bidders have independent signals, the K-dimensional GSP auction, the K-dimensional VCG auction and the Generalized English Auction always implement the same allocation and yield zero expected payoff to the bidder with the lowest signal. The revenue equivalence follows as a result.

6.2 Revenue Comparison with the Optimal Position Auction

I next characterize the optimal position auction under interdependent values subject to no reserve price[14] and then compare expected revenues of the K-dimensional GSP auction, the K-dimensional VCG auction and the Generalized English Auction to the optimal revenue implementable in position auctions subject to no reserve price.

6.2.1 Mechanism Design and Solution Concepts

Under the revelation principle, I characterize the optimal position auction as a direct mechanism, in which bidders report private signals directly, while the value function $v(x_i, x_{-i})$ and signal distribution $F(x_1, x_2, \cdots, x_N)$ are common knowledge. To make the expected revenue of the optimal position auction comparable to expected

[14] Ulku (2013)[37] characterizes the optimal mechanism for allocating a set of heterogeneous items under interdependent values. The optimal position auction characterized in this paper can be viewed as a corollary of Ulku (2013)[37] in the special environment of position auctions. The objective is to compare the revenues of the three efficient auctions to the optimal revenue.

revenues of the three practical auctions analyzed in section 5, I restrict attention to the optimal position auction subject to no reserve price. A position auction mechanism (μ, p) consists of an allocation rule $\mu_i(x)$ and a payment rule $p_i(x)$ for every bidder i, where $\mu_i(x) = \left(\mu_i^{(1)}(x), \mu_i^{(2)}(x), \cdots, \mu_i^{(K)}(x)\right)$ is the vector of probabilities that bidder i wins position $1, 2, \cdots, K$ given reported signals $x \in [0, \bar{x}]^N$, and $p_i(x)$ is the expected payment of bidder i given reported signals $x \in [0, \bar{x}]^N$. In a deterministic mechanism, $\mu_i^{(k)}(x) \in \{0, 1\}$ for all k and $p_i(x)$ is the actual payment.

Given a CTR profile $(\alpha_1, \alpha_2, \cdots, \alpha_K)$ and allocation rule μ, the expected click-through rate q_i assigned to bidder i under report $x = (x_1, x_2, \cdots, x_N)$ is given by

$$q_i(x) = \sum_{k=1}^{K} \alpha_k \mu_i^{(k)}(x) \tag{30}$$

For notational simplicity, I use the expected CTR $\left\{q_i(x)\right\}_{i=1}^{N}$ instead of N vectors of expected probabilities $\left\{\left(\mu_i^{(1)}(x), \mu_i^{(2)}(x), \cdots, \mu_i^{(K)}(x)\right)\right\}_{i=1}^{N}$ as the allocation rule in the analysis. I use (q, p) and (μ, p) to refer to the same mechanism interchangeably if $q_i(x) = \sum_{k=1}^{K} \alpha_k \mu_i^{(k)}(x)$. The feasibility condition of the allocation rule in a position auction mechanism is defined below:

Definition 9. *An allocation rule in the form of $\mu(x)$ is feasible if*

$$0 \leq \sum_{i=1}^{N} \mu_i^{(k)}(x) \leq 1, \quad \forall\ k, \quad \textit{and} \quad 0 \leq \sum_{k=1}^{K} \mu_i^{(k)}(x) \leq 1, \quad \forall\ i \tag{31}$$

An allocation rule in the form of $q(x)$ is feasible if $q_i(x) = \sum_{k=1}^{K} \alpha_k \mu_i^{(k)}(x)$ for all i for some allocation rule $\mu(x)$ satisfying condition (31).

For any bidder i with signal x_i, the interim utility $U_i(x_i)$ is given by

$$U_i(x_i) = \int_{x_{-i}} \Big[q_i(x_i, x_{-i}) v_i(x_i, x_{-i}) - p_i(x_i, x_{-i})\Big] f_{-i|i}(x_{-i}|x_i) dx_{-i} \tag{32}$$

where $u_i(x_i, x_{-i}) = q_i(x_i, x_{-i}) v_i(x_i, x_{-i}) - p_i(x_i, x_{-i})$ is the ex-post utility of bidder i

given the signal profile (x_i, x_{-i}). I now give the definition of two solution concepts:

Definition 10. *A position auction mechanism* (q, p) *is Bayesian incentive compatible (IC) and individually rational (IR) if for every bidder* i*, for any true signal* x_i *and any report* $x_i^{'}$*,*

$$\begin{aligned} U_i(x_i) &\geq \int_{x_{-i}} \left[q_i(x_i^{'}, x_{-i}) v_i(x_i, x_{-i}) - p_i(x_i^{'}, x_{-i}) \right] f_{-i|i}(x_{-i}|x_i) dx_{-i} \\ U_i(x_i) &\geq 0 \end{aligned} \tag{33}$$

Definition 11. *A position auction mechanism* (q, p) *is ex-post incentive compatible (IC) and individually rational (IR) if for every bidder* i*, for any true signal profile* (x_i, x_{-i}) *and any report* $x_i^{'}$*,*

$$\begin{aligned} u_i(x_i, x_{-i}) &\geq q_i(x_i^{'}, x_{-i}) v_i(x_i, x_{-i}) - p_i(x_i^{'}, x_{-i}) \\ u_i(x_i, x_{-i}) &\geq 0 \end{aligned} \tag{34}$$

6.2.2 Characterization of the Optimal Position Auction

Given any profile of signals, define bidder i's marginal revenue $MR_i(x_i, x_{-i})$ as

$$MR_i(x_i, x_{-i}) = v_i(x_i, x_{-i}) - \frac{1 - F_i(x_i|x_{-i})}{f_i(x_i|x_{-i})} \times \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \tag{35}$$

For any bidder i, given a vector of opponents' reported signals x_{-i}, define $\hat{X}^k(x_{-i})$ as the minimum value that bidder i's signal can take such that bidder i has the k-th highest MR_i among all bidders:

$$\hat{X}^k(x_{-i}) = \inf \left\{ x_i \;\middle|\; MR_i(x_i, x_{-i}) \geq kmax_{j \neq i} \{ MR_j(x_j, x_i, x_{-ij}) \} \right\} \tag{36}$$

in which $kmax_{j \neq i}\{MR_j(x_j, x_i, x_{-ij})\}$ is value of the k-th highest marginal revenue among bidder i's opponents given report x, and x_{-ij} is the vector of signals reported by bidders other than i and j. The following two regularity conditions are provided such that the optimal position auction assigns positions in the rank ordering of $MR_i(x_i, x_{-i})$

given report (x_i, x_{-i}).

R1 (*Value Regularity*): Given any profile of signals x, for any two bidders i, j,

$$\text{If} \quad x_i > x_j, \quad \text{then} \quad v_i(x_i, x_j, x_{-ij}) > v_j(x_j, x_i, x_{-ij}) \tag{37}$$

Note that **R1** is directly implied by assumptions **A1-A3**.

R2 (*MR Monotonicity*): Given any report of signals x, for all bidder i,

$$\frac{\partial MR_i(x_i, x_{-i})}{\partial x_i} > 0, \quad \forall x_{-i} \tag{38}$$

The next proposition characterizes the optimal position auction subject to no reserve price among all ex-post IC and IR mechanisms under **R1** and **R2**.

Proposition 7. *Under regularity conditions **R1** and **R2**, suppose the expected CTR is given by*

$$q_i^*(x_i, x_{-i}) = \begin{cases} \alpha_k & \text{if} \quad \hat{X}^k(x_{-i}) \leq x_i < \hat{X}^{k-1}(x_{-i}) \\ 0 & \text{if} \quad x_i < \hat{X}^K(x_{-i}) \end{cases} \tag{39}$$

Any tie is broken randomly. Suppose also that the payment rule is given by

$$p_i^*(x_i, x_{-i}) = q_i(x_i, x_{-i}) v_i(x_i, x_{-i}) - \int_0^{x_i} q_i(s, x_{-i}) \frac{\partial v_i(s, x_{-i})}{\partial s} ds \tag{40}$$

Then (q^*, p^*) is an optimal position auction among all the ex-post IC and IR mechanisms subject to no reserve price.

Proof. See Appendix. □

Note that in the special case of independent signals, each bidder's marginal revenue is given by

$$MR_i(x_i, x_{-i}) = v_i(x_i, x_{-i}) - \frac{1 - F_i(x_i)}{f_i(x_i)} \times \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \tag{41}$$

as $F_i(x_i|x_{-i}) = F_i(x_i)$ and $f_i(x_i|x_{-i}) = f_i(x_i)$ under independent signals. The next proposition shows that under **R1** and **R2**, conditional on having no reserve price, the optimal position auction (q^*, p^*) characterized in proposition 7 is also optimal among all Bayesian IC and IR mechanisms when signals are independent.

Proposition 8. *Under regularity conditions* ***R1*** *and* ***R2****, if signals are independent, then* (q^*, p^*) *is an optimal position auction among all the Bayesian IC and IR mechanisms subject to no reserve price.*

Proof. See Appendix. □

Since all ex-post IC and IR mechanisms are also Bayesian IC and IR mechanisms, the optimality of (q^*, p^*) under independent signals is stronger.

6.2.3 Revenue Comparison with the Optimal Position Auction

Ausubel (1999)[5] proposes a direct revelation mechanism called the *generalized Vickrey auction* for selling homogeneous items efficiently under interdependent values. I next show the optimal position auction subject to no reserve price characterized in Proposition 7 can be viewed as a generalized Vickrey auction under the context of vertically differentiated items with single-demand bidders. Then I will compare the expected revenue of the Generalized-VCG mechanism to the expected revenues of the GEA, the K-dimensional GSP auction and the K-dimensional VCG auction.

For an arbitrary bidder i, given a vector of opponents' bids x_{-i}, let $\hat{x}^k(x_{-i})$ be the minimum value that bidder i's signal can take such that bidder i has at least the k-th highest value among all bidders:

$$\hat{x}^k(x_{-i}) = \inf \left\{ x_i \;\middle|\; v_i(x_i, x_{-i}) \geq kmax_{j \neq i}\big\{v_j(x_j, x_i, x_{-ij})\big\} \right\} \tag{42}$$

in which $kmax_{j \neq i}\big\{v_j(x_j, x_i, x_{-ij})\big\}$ is the k-th highest value received by bidder i's opponents given report x, and x_{-ij} is the vector of signals reported by bidders other than i and j.

Definition 12. *Under the context of position auctions, the generalized Vickrey auction is defined as follows:*

$$q_i^V(x_i, x_{-i}) = \begin{cases} \alpha_k & \textit{if} \quad \hat{x}^k(x_{-i}) \leq x_i < \hat{x}^{k-1}(x_{-i}) \\ 0 & \textit{if} \quad x_i < \hat{x}^K(x_{-i}) \end{cases}$$

$$p_i^V(x_i, x_{-i}) = \begin{cases} \sum_{j=k}^{K}(\alpha_j - \alpha_{j+1})v_i(\hat{x}^j(x_{-i}), x_{-i}) & \textit{if} \quad \hat{x}^k(x_{-i}) \leq x_i < \hat{x}^{k-1}(x_{-i}) \\ 0 & \textit{if} \quad x_i < \hat{x}^K(x_{-i}) \end{cases} \tag{43}$$

Any tie is broken randomly.

The next proposition shows that the optimal position auction subject to no reserve price (q^*, p^*) among all ex-post IC and IR mechanisms is equivalent to the generalized Vickrey auction when an additional regularity condition described below is satisfied:

R3 (*MR regularity*): For all i, j, given any report x,

$$\text{if} \quad x_i > x_j, \quad \text{then} \quad MR_i(x_i, x_j, x_{-ij}) > MR_j(x_j, x_i, x_{-ij}) \tag{44}$$

Proposition 9. *Under regularity conditions* ***R1****,* ***R2*** *and* ***R3****, the optimal position auction subject to no reserve price among all ex-post IC and IR mechanisms is equivalent to the generalized Vickrey auction. This optimal revenue is weakly higher than the expected revenue of the GEA, which is in turn weakly higher than the expected revenues of the K-dimensional GSP auction and the K-dimensional VCG auction.*

Proof. See Appendix. □

The intuition behind the revenue ranking in Proposition 9 comes from that in the generalized Vickrey auction, the payment of each bidder depends on the entire reported signal profile from opposing bidders, while the payment of each bidder only depends on a subset of opponents' signals in the GEA, and depends on none of opponents' signals in the K-dimensional GSP auction and the K-dimensional VCG auction. Under the logic of the Linkage Principle, when signals are affiliated, the expected revenue of an auction is greater when each bidder's payment depends on more of its opponents' signals. The revenue dominance of the generalized Vickrey auction comes from the additional assumption that the auctioneer must know the value function $v_i(x_i, x_{-i})$. As pointed out by Ausubel and Cramton (2004)[7], this assumption is usually impractical. In practical auctions where the auctioneer does not know bidders' value functions, it is natural to expect lower revenues.

On the other hand, revenue equivalence holds among the generalized Vickrey auction, the GEA, the K-dimensional GSP auction and the K-dimensional VCG auction under independent signals:

Corollary 5. *When bidders have independent signals, under regularity conditions* ***R1****,* ***R2*** *and* ***R3****, the optimal position auction subject to no reserve price among all Bayesian IC and IR mechanisms is equivalent to the generalized Vickrey auction. Moreover, this optimal revenue can be practically implemented by the GEA, the K-dimensional GSP auction, and the K-dimensional VCG auction.*

The main insight from Corollary 5 is that under independent signals and regularity conditions **R1**, **R2** and **R3**, the three K-dimensional position auctions dominate the one-dimensional position auctions in both efficiency and revenue.

7 Conclusions

This paper extends the study of position auctions into the information structure of interdependent values, which better captures the oligopoly relationship among advertisers and the uncertainty all advertisers face under continuous demand shocks. I show that the commonly-used GSP auctions and VCG-like auctions can be inefficient under this information structure. On the other hand, the Generalized English Auction introduced by Edelman et al. (2007)[15] as a theoretical modeling tool still retains full efficiency. Although it is not feasible to implement the Generalized English Auction in practice, I show that efficiency can be restored in the GSP auction and the VCG-like auction by adopting a multi-dimensional bidding language that allows bidders to condition their bid on position. Moreover, I show that the modified K-dimensional GSP auction and the K-dimensional VCG auction are revenue equivalent, while the Generalized English Auction yields higher revenue under linkage principle. With independent signals, the K-dimensional GSP and VCG auctions can also implement the optimal revenue subject to no reserve price.

The inefficiency of the GSP auction provides a sharp contrast to previous studies that favor the GSP auction under complete information. Moreover, it also extends Gomes and Sweeney (2014)[21]'s result into a broader information structure with new insights. While the inefficiency result in Gomes and Sweeney (2014)[21] implies that the oversimplified payment rule in the GSP auction can be problematic under pure private values, this paper brings some new insights by showing that the oversimplified bidding language is a more intrinsic problem under every setting when bidders might prefer winning a lower ranked position to a higher ranked position, and this problem exists not only in the GSP auction, but also in the VCG-like auction under interdependent values. The inefficiency of the VCG-like auction provides a contrast to Milgrom and Weber (1982)[31], who show that the second-price single-unit auction is still efficient under interdependent values. This is because bidders have differentiated bid-shading incentives only when there are multiple differentiated positions. With one-dimensional bidding language, those bidders with higher signals have stronger bid-shading incentives, as they are more likely to win the highest ranked position. By allowing bidders to condition their bids on positions, bidders do not need to worry about winning a higher ranked position when submitting their bids for lower ranked positions. The differentiated bid-shading incentive across bidders is therefore replaced by differentiated bid-shading incentives across positions. A bidder with higher signal always bids higher than a bidder with lower signal for every position, and efficiency follows as a result. On the other hand, the Generalized English Auction also implicitly adopts a multi-dimensional bidding language, as it allows bidders to update their strategy within the dynamic process. The revenue dominance of the Generalized English Auction is a generalization of Milgrom and Weber (1982)[31]'s revenue ranking between English auctions and second-price auctions. The revenue equivalence between the K-dimensional GSP and K-dimensional VCG auctions contribute to the previous literature that compares the GSP auction and the VCG auction under complete information. Putting together these insights, we can conclude that under interdependent values, adopting a multi-dimensional bidding language that allows bidders to bid differently across positions is essential in practical position auctions under both efficiency-maximizing and revenue-maximizing objectives. This conclusion implies that there exists a trade-off between simplicity versus efficiency and revenue in auction design. Moreover, comparing the equilibrium of K-dimensional auctions to their one-dimensional counterparts implies that there also exists a trade-off between simplicity in auction design and simplicity in bidding strategy. These implications contribute to the discussion on the cost of simplicity in mechanism design in both economics and computer science literature.

In addition to its theoretical contributions, this paper has some practical contribution by proposing two practical auction formats for allocating sponsored advertising spaces on a wide range of online platforms, including search engines such as Google and Yahoo!, online shopping platforms such as Amazon and eBay, online rating and booking platforms such as Yelp and TripAdvisor, and social media such as Facebook, Twitter, and Instagram. All of these two-sided platforms share the common characteristics that advertisers competing for the same advertising space are likely selling substitutable products or services and therefore are subject to demand shocks in the same market. When interdependency is likely to present in bidders' values, it may worth to use the more complicated, multi-dimensional bidding language in GSP and VCG auctions in order to guarantee efficiency and improve revenue.

This paper points to two future research directions. First, this paper follows previous literature on position auctions and assumes bidders have single-unit demands. However, bidders may have multi-unit demands in real position auctions. For example, an advertiser may demand consecutive slots on the first search result page or demand a slot on each of the first three search result pages under a keyword. One natural extension of this paper is to allow bidders to have multi-unit demands and explore how to design efficient and optimal position auctions with multi-unit demand bidders. Second, it would be interesting to conduct an experimental study to test the theoretical predictions in this paper and quantify the efficiency and revenue changes that result from adopting a multi-dimensional bidding language in GSP and VCG-like auctions.

Appendix

Proof of Lemma 1:

Proof. I first show that if an equilibrium bidding strategy in a one-dimensional assortative position auction is symmetric and strictly increasing, then the equilibrium must be efficient. Let $\beta(x)$ denote the equilibrium bidding strategy. $\beta^{'}(x) > 0$ implies that $\beta(x_i) > \beta(x_j)$ for any $x_i > x_j$. Under the assortative ranking rule, bidder i is placed above bidder j if $x_i > x_j$, so the equilibrium allocation must be efficient.

I next show that if an equilibrium of a one-dimensional assortative position auction is efficient, then the equilibrium bidding strategy must be strictly increasing and symmetric across bidders. Suppose an efficient equilibrium $\big(\beta_1(x_1), \beta_2(x_2), \cdots, \beta_N(x_N)\big)$ exists in a one-dimensional assortative position auction, then a bidder who receives signal x_i must be placed above a bidder who receives a lower signal $x_j < x_i$ if both win some position in equilibrium. For an arbitrary bidder i, take any value $x_i^{'} > x_i$, then there is positive probability that some of bidder i's opponents receive signals between x_i and $x_i^{'}$, i.e., there exists $j \neq i$ with signal $x_j \in (x_i, x_i^{'})$. Efficiency requires that j is placed below i when bidder i receives $x_i^{'}$, and j is placed above i when i receives x_i. Under the assortative ranking rule, this requires

$$\begin{aligned} \beta_j(x_j) &< \beta_i(x_i^{'}) \\ \beta_j(x_j) &> \beta_i(x_i) \end{aligned} \tag{45}$$

Suppose $\beta_i(x_i^{'}) \leq \beta_i(x_i)$, then for any value of $\beta_j(x_j)$, it is impossible for condition (45) to hold, which yields a contradiction. Therefore, at any efficient equilibrium, bidder i must bid strictly higher when receiving signal $x_i^{'}$ than receiving signal x_i, i.e., for every bidder i, we must have

$$x_i^{'} > x_i \quad \rightarrow \quad \beta_i(x_i^{'}) > \beta_i(x_i) \tag{46}$$

Therefore, every bidder must use a strictly increasing bidding strategy in an efficient equilibrium, so $\beta_i^{'}(x_i) > 0$ for all i.

Next, suppose there exists an efficient equilibrium that is not symmetric, i.e., there exists $i \neq j$ s.t. $\beta_i(\hat{x}) \neq \beta_j(\hat{x})$ at some $\hat{x} \in [0, \bar{x}]$. Without loss of generality, assume $\beta_i(\hat{x}) < \beta_j(\hat{x})$ for some $\hat{x} \in [0, \bar{x}]$. Since $\beta_i(.)$ and $\beta_j(.)$ are continuous, there exists some $x_i, x_j \in [0, \bar{x}]$ s.t. $x_j < \hat{x} < x_i$, but $\beta_i(x_i) < \beta_j(x_j)$. Under the assortative ranking rule, this means that bidder j who receives the lower signal x_j will get a higher position than bidder i who receives the higher signal $x_i > x_j$, which contradicts the efficiency assumption. Therefore, if an efficient equilibrium exists in a one-dimensional assortative position auction, then the equilibrium bidding strategy must be symmetric across bidders, i.e., $\beta_i(.) = \beta(.)$ for all i.

□

Proof of Lemma 2:

Proof. Define $v^{1,2}(x_i, y_1, y_2)$ as bidder i's expected value per click conditional on her own signal equals to x_i, the highest signal among her opponents Y_1 equals to y_1, the second highest signal among her opponents Y_2 equals to y_2:

$$v^{1,2}(x_i, y_1, y_2) = E\big[v_i \big| X = x_i, Y_1 = y_1, Y_2 = y_2\big] \tag{47}$$

Suppose a monotonic Bayesian equilibrium bidding strategy $\beta(.)$ exists. For any arbitrary bidder i, suppose all of i's opposing bidders follow the monotonic Bayesian equilibrium bidding strategy $\beta(.)$. Let $\beta^{-1}(.)$ denote the inverse function of $\beta(.)$. Then bidder i's best response bid b_i^* maximizes

$$\begin{aligned}\Pi(b_i|x_i) = &\int_0^{\beta^{-1}(b_i)} \int_0^{y_1} \alpha_1 \Big[v^{1,2}(x_i, y_1, y_2) - \beta(y_1)\Big] g_i^{2,1}(y_2, y_1|x_i) dy_2 dy_1 \\ &+ \int_{\beta^{-1}(b_i)}^{\bar{x}} \int_0^{\beta^{-1}(b_i)} \alpha_2 \Big[v^{1,2}(x_i, y_1, y_2) - \beta(y_2)\Big] g_i^{2,1}(y_2, y_1|x_i) dy_2 dy_1\end{aligned} \tag{48}$$

in which $g_i^{2,1}(y_2, y_1|x_i)$ is the conditional joint density function of (Y_2, Y_1) given $X = x_i$. Let $g_{1|2}(y_1|y_2, x_i)$ and $g_{2|1}(y_2|y_1, x_i)$ be conditional marginal densities of Y_1 given (Y_2, X) and Y_2 given (Y_1, X) respectively. Let $g_1(y_1|x_i)$ and $g_2(y_2|x_i)$ be conditional marginal densities of Y_1 and Y_2 given $X = x_i$ respectively, then $g_i^{2,1}(y_2, y_1|x_i) = g_{1|2}(y_1|y_2, x_i)g_2(y_2|x_i) = g_{2|1}(y_2|y_1, x_i)g_1(y_1|x_i)$.

Take derivative of the objective function (48) with respect to b_i:

$$
\begin{aligned}
\frac{d\Pi(b_i|x_i)}{db_i} =& \frac{g_1(\beta^{-1}(b_i)|x_i)}{\beta'(\beta^{-1}(b_i))}\int_0^{\beta^{-1}(b_i)} \alpha_1\Big[v^{1,2}(x_i, \beta^{-1}(b_i), y_2) - b_i\Big]g_{2|1}\big(y_2\big|\beta^{-1}(b_i), x_i\big)dy_2 \\
&- \frac{g_1(\beta^{-1}(b_i)|x_i)}{\beta'(\beta^{-1}(b_i))}\int_0^{\beta^{-1}(b_i)} \alpha_2\Big[v^{1,2}(x_i, \beta^{-1}(b_i), y_2) - \beta(y_2)\Big]g_{2|1}\big(y_2\big|\beta^{-1}(b_i), x_i\big)dy_2 \\
&+ \frac{g_2(\beta^{-1}(b_i)|x_i)}{\beta'(\beta^{-1}(b_i))}\int_{\beta^{-1}(b_i)}^{\bar{x}} \alpha_2\Big[v^{1,2}(x_i, y_1, \beta^{-1}(b_i)) - b_i\Big]g_{1|2}\big(y_1\big|\beta^{-1}(b_i), x_i\big)dy_1
\end{aligned} \tag{49}
$$

Since $\beta(x_i)$ is an equilibrium, it is optimal for bidder i to bid $b_i^* = \beta(x_i)$ when her opponents follow $\beta(.)$. Evaluate $\frac{d\Pi(b_i)}{db_i}$ at $b_i^* = \beta(x_i)$:

$$
\begin{aligned}
\frac{d\Pi(\beta(x_i)|x_i)}{db_i} =& \frac{g_1(x_i|x_i)}{\beta'(x_i)}\int_0^{x_i} \alpha_1\Big[v^{1,2}(x_i, x_i, y_2) - \beta(x_i)\Big]g_{2|1}(y_2|x_i, x_i)dy_2 \\
&- \frac{g_1(x_i|x_i)}{\beta'(x_i)}\int_0^{x_i} \alpha_2\Big[v^{1,2}(x_i, x_i, y_2) - \beta(y_2)\Big]g_{2|1}(y_2|x_i, x_i)dy_2 \\
&+ \frac{g_2(x_i|x_i)}{\beta'(x_i)}\int_{x_i}^{\bar{x}} \alpha_2\Big[v^{1,2}(x_i, y_1, x_i) - \beta(x_i)\Big]g_{1|2}(y_1|x_i, x_i)dy_1
\end{aligned} \tag{50}
$$

According to the definition of $v^1(x_i, x_i)$ and $v^2(x_i, x_i)$, equation (50) is equivalent to

$$
\begin{aligned}
\frac{d\Pi(\beta(x_i)|x_i)}{db_i} =& \frac{g_1(x_i|x_i)}{\beta'(x_i)}\alpha_1\Big[v^1(x_i, x_i) - \beta(x_i)\Big] \\
&- \frac{g_1(x_i|x_i)}{\beta'(x_i)}\alpha_2\Big[v^1(x_i, x_i) - \int_0^{x_i} \beta(y_2)g_{2|1}(y_2|x_i, x_i)dy_2\Big] \\
&+ \frac{g_2(x_i|x_i)}{\beta'(x_i)}\alpha_2\Big[v^2(x_i, x_i) - \beta(x_i)\Big]
\end{aligned} \tag{51}
$$

Bidding $b_i^* = \beta(x_i)$ maximizes $\Pi(b_i|x_i)$ only if $\frac{d\Pi(\beta(x_i)|x_i)}{db_i} = 0$. Setting equation (51) to zero and rearrange yields

$$\beta(x_i) = \frac{g_1(x_i|x_i)\Big[(\alpha_1-\alpha_2)v^1(x_i,x_i)+\alpha_2\int_0^{x_i}\beta(y_2)g_{2|1}(y_2|x_i,x_i)dy_2\Big]+g_2(x_i|x_i)\alpha_2 v^2(x_i,x_i)}{\alpha_1 g_1(x_i|x_i)+\alpha_2 g_2(x_i|x_i)} \tag{52}$$

This is a Volterra equation of the second kind. In the one-dimensional GSP Auction with 2 positions, if a monotonic equilibrium bidding strategy $\beta^G(x_i)$ exists, then $\beta^G(x_i)$ must satisfy the Volterra equation (52) for all $x_i \in [0, \bar{x}]$.

□

Proof of Lemma 3:

Proof. Suppose a monotonic symmetric Bayesian equilibrium bidding strategy $\beta(.)$ exists in the one-dimensional VCG auction. For an arbitrary bidder i, suppose all of i's opposing bidders follow the monotonic Bayesian equilibrium bidding strategy $\beta(.)$. Let $\beta^{-1}(.)$ denote the inverse function of $\beta(.)$. Then bidder i's best response bid b_i^* maximizes

$$\begin{aligned}\Pi(b_i|x_i) = &\int_0^{\beta^{-1}(b_i)}\int_0^{y_1}\Big\{\alpha_1\big[v^{1,2}(x_i,y_1,y_2)-\beta(y_1)\big]+\alpha_2\big[\beta(y_1)-\beta(y_2)\big]\Big\}g_i^{2,1}(y_2,y_1|x_i)dy_2dy_1\\ &+\int_{\beta^{-1}(b_i)}^{\bar{x}}\int_0^{\beta^{-1}(b_i)}\alpha_2\big[v^{1,2}(x_i,y_1,y_2)-\beta(y_2)\big]g_i^{2,1}(y_2,y_1|x_i)dy_2dy_1\end{aligned} \tag{53}$$

Take derivative with respect to b_i:

$$\begin{aligned}\frac{d\Pi(b_i|x_i)}{db_i} =& \frac{g_1(\beta^{-1}(b_i)|x_i)}{\beta'(\beta^{-1}(b_i))}\int_0^{\beta^{-1}(b_i)}(\alpha_1-\alpha_2)\Big[v^{1,2}(x_i,\beta^{-1}(b_i),y_2)-b_i\Big]g_{2|1}\big(y_2\big|\beta^{-1}(b_i),x_i\big)dy_2\\ &+\frac{g_2(\beta^{-1}(b_i)|x_i)}{\beta'(\beta^{-1}(b_i))}\int_{\beta^{-1}(b_i)}^{\bar{x}}\alpha_2\Big[v^{1,2}(x_i,y_1,\beta^{-1}(b_i))-b_i\Big]g_{1|2}\big(y_1\big|\beta^{-1}(b_i),x_i\big)dy_1\end{aligned} \tag{54}$$

Since $\beta(x_i)$ is an equilibrium, $b_i^* = \beta(x_i)$ maximizes $\Pi(b_i|x_i)$ for any value of x_i.

For all $x_i \in [0, \bar{x}]$, evaluate $\frac{d\Pi(b_i|x_i)}{db_i}$ at $b_i^* = \beta(x_i)$ gives

$$\begin{aligned}\frac{d\Pi(\beta(x_i)|x_i)}{db_i} =& \frac{g_1(x_i|x_i)}{\beta'(x_i)} \int_0^{x_i} (\alpha_1 - \alpha_2)\Big[v^{1,2}(x_i, x_i, y_2) - \beta(x_i)\Big] g_{2|1}\big(y_2\big|x_i, x_i\big) dy_2 \\ &+ \frac{g_2(x_i|x_i)}{\beta'(x_i)} \int_{x_i}^{\bar{x}} \alpha_2 \Big[v^{1,2}(x_i, y_1, x_i) - \beta(x_i)\Big] g_{1|2}\big(y_1\big|x_i, x_i\big) dy_1\end{aligned} \tag{55}$$

According to the definition of $v^1(x_i, x_i)$ and $v^2(x_i, x_i)$, this is equivalent to

$$\frac{d\Pi(\beta(x_i)|x_i)}{db_i} = \frac{g_1(x_i|x_i)}{\beta'(x_i)}(\alpha_1 - \alpha_2)\big[v^1(x_i, x_i) - \beta(x_i)\big] + \frac{g_2(x_i|x_i)}{\beta'(x_i)}\alpha_2\big[v^2(x_i, x_i) - \beta(x_i)\big] \tag{56}$$

Bidding $\beta(x_i)$ maximizes $\Pi(b_i|x_i)$ only if $\frac{d\Pi(\beta(x_i)|x_i)}{db_i} = 0$, which means that bidder i cannot increase $\Pi(b_i|x_i)$ by increasing or decreasing bid from $\beta(x_i)$ by any small amount. Set $\frac{d\Pi(\beta(x_i)|x_i)}{db_i} = 0$ and rearrange the equation yields

$$\beta(x_i) = \frac{g_1(x_i|x_i)(\alpha_1 - \alpha_2)v^1(x_i, x_i) + g_2(x_i|x_i)\alpha_2 v^2(x_i, x_i)}{g_1(x_i|x_i)(\alpha_1 - \alpha_2) + g_2(x_i|x_i)\alpha_2} \tag{57}$$

which characterizes the unique equilibrium bidding strategy $\beta^V(x_i)$ in the one-dimensional VCG auction. □

Proof of Proposition 1:

Proof. Suppose the unique equilibrium bidding strategy $\beta^G(x_i)$ characterized in Lemma 2 is continuous and strictly increasing in x_i given any CTR profile (α_1, α_2). First observe that since $\beta^G(.)$ is continuous, when x_i approaches $\bar{x}$, the equilibrium bid $\beta^G(x_i)$ characterized in Lemma 2 approaches $\beta^G(\bar{x})$:

$$\lim_{x_i \to \bar{x}} \beta^G(x_i) = \beta^G(\bar{x}) = \Big(\frac{\alpha_1 - \alpha_2}{\alpha_1}\Big) v^1(\bar{x}, \bar{x}) + \Big(\frac{\alpha_2}{\alpha_1}\Big) \int_0^{\bar{x}} \beta(y_2) g_{2|1}(y_2|\bar{x}, \bar{x}) dy_2 \tag{58}$$

Consider the case when α_2 approaches α_1, the equilibrium bid $\beta^G(\bar{x})$ approaches

$$\lim_{\alpha_2 \to \alpha_1} \beta^G(\bar{x}) = \int_0^{\bar{x}} \beta(y_2) g_{2|1}(y_2|\bar{x}, \bar{x}) dy_2 \tag{59}$$

which implies that $\beta^G(\bar{x})$ satisfies

$$\lim_{\alpha_2 \to \alpha_1} \int_0^{\bar{x}} \Big(\beta^G(\bar{x}) - \beta^G(y_2)\Big) g_{2|1}(y_2|\bar{x}, \bar{x}) dy_2 = 0 \tag{60}$$

However, equation (60) yields a contradiction to the assumption that $\beta^G(x_i)$ is strictly increasing in x_i, since for any strictly increasing function, $\beta^G(\bar{x}) > \beta^G(y_2)$ for any $0 \leq y_2 < \bar{x}$ and $\beta^G(\bar{x}) = \beta^G(y_2)$ at $y_2 = \bar{x}$. Therefore, it is impossible for any strictly increasing $\beta^G(x_i)$ to satisfy equation (60) at $x_i = \bar{x}$. Since $\beta^G(x_i)$ approaches $\beta^G(\bar{x})$ when x_i approaches $\bar{x}$, this contradiction also applies to any x_i sufficiently close to $\bar{x}$. Therefore, it is impossible for the equilibrium $\beta^G(x_i)$ characterized by Lemma 2 to be strictly increasing under every CTR profile. Since $\beta^G(x_i)$ is the unique equilibrium bidding strategy, there exists no monotonic equilibrium in the one-dimensional GSP auction with two positions under some CTR profile. Given the result of Lemma 1, this implies that there exists some number of positions K with some CTR profile such that no efficient equilibrium exists in the one-dimensional GSP auction. □

Proof of Proposition 2:

Proof. Define $\gamma(x_i; \alpha_1, \alpha_2)$ as the weighting factor in the equilibrium bidding function $\beta^V(x_i)$ characterized in Lemma 3:

$$\gamma(x_i; \alpha_1, \alpha_2) = \frac{g_1(x_i|x_i)(\alpha_1 - \alpha_2)}{g_1(x_i|x_i)(\alpha_1 - \alpha_2) + g_2(x_i|x_i)\alpha_2} \tag{61}$$

then the equilibrium bidding strategy characterized in Lemma 3 can be rewritten as

$$\beta^V(x_i) = \gamma(x_i; \alpha_1, \alpha_2) v^1(x_i, x_i) + \Big(1 - \gamma(x_i; \alpha_1, \alpha_2)\Big) v^2(x_i, x_i) \tag{62}$$

Take derivative of $\beta(x_i) = \gamma(x_i)v^1(x_i, x_i) + (1-\gamma(x_i))v^2(x_i, x_i)$ with respect to x_i:

$$\frac{d\beta^V(x_i)}{dx_i} = \underbrace{\gamma(x_i)\Big[\frac{\partial v^1(x_i,x_i)}{\partial x_i}\Big] + (1-\gamma(x_i))\Big[\frac{\partial v^2(x_i,x_i)}{\partial x_i}\Big]}_{\text{bid-increasing incentive from higher expected values}} + \underbrace{\frac{\partial \gamma(x_i)}{\partial x_i}\Big[v^1(x_i,x_i) - v^2(x_i,x_i)\Big]}_{\text{bid-shading incentive from the ``winner's curse''}} \tag{63}$$

The first two terms in equation (63) capture the positive effect of greater expected values on $\beta^V(x_i)$ when x_i increases. As x_i increases, the expected values conditional on winning both position 1 and position 2 increase, which causes equilibrium bid $\beta^V(x_i)$ to increase. The last term captures the negative effect of the "winner's curse" on $\beta^V(x_i)$. As x_i increases, bidder i is more likely to win the first position at any monotonic equilibrium, which amplifies the "winner's curse." When the negative effect from the "winner's curse" dominates the positive effect from increased expected values, the sign of $\frac{d\beta^V(x_i)}{dx_i}$ can be negative.

Note that given any CTR profile (α_1, α_2), for any $x_i \in [0, \bar{x}]$, the magnitude of the "winner's curse," $v^1(x_i, x_i) - v^2(x_i, x_i)$, is multiplied by $\frac{\partial\gamma(x_i)}{\partial x_i}$. The later can be expressed as

$$\frac{\partial \gamma(x_i;\alpha_1,\alpha_2)}{\partial x_i} = \frac{(\alpha_1-\alpha_2)\alpha_2\left[\frac{\partial g_1(x_i|x_i)}{\partial x_i}g_2(x_i|x_i) - g_1(x_i|x_i)\frac{\partial g_2(x_i|x_i)}{\partial x_i}\right]}{\Big[g_1(x_i|x_i)(\alpha_1-\alpha_2) + g_2(x_i|x_i)\alpha_2\Big]^2} > 0 \tag{64}$$

For any CTR profile (α_1, α_2) satisfying $0 < \alpha_2 < \alpha_1$, take limit of $\frac{\partial\gamma(x_i;\alpha_1,\alpha_2)}{\partial x_i}$ when x_i approaches $\bar{x}$ yields

$$\lim_{x_i\to\bar{x}} \frac{\partial \gamma(x_i;\alpha_1,\alpha_2)}{\partial x_i} = -\Big(\frac{\alpha_2}{\alpha_1-\alpha_2}\Big) \times \frac{1}{g_1(\bar{x}|\bar{x})} \times \frac{\partial g_2(\bar{x}|\bar{x})}{\partial x_i} \tag{65}$$

When α_2 is sufficiently close to α_1, the denominator becomes sufficiently close to 0 so that $\lim_{x_i\to\bar{x}} \frac{\partial\gamma(x_i;\alpha_1,\alpha_2)}{\partial x_i}$ approaches infinity. As long as $v^1(x_i, x_i) < v^2(x_i, x_i)$, the negative impact from "winner's curse" will be dominant when x_i is sufficiently close to

$\bar{x}$ and α_2 is sufficiently close to α_1. Therefore, under any non-trivially interdependent values, when there are $K = 2$ positions, there always exists some CTR profile (α_1, α_2) in which α_2 is strictly lower than but sufficiently close to α_1 s.t. the equilibrium bid $\beta^V(x_i)$ is decreasing in x_i for values of x_i close to the upper boundary $\bar{x}$. This demonstrates that there always exists some number of positions K with some CTR profile such that no efficient equilibrium exists in the one-dimensional VCG auction. □

Proof of Lemma 4:

Proof. It is straightforward to see that if every bidder adopts a symmetric and strictly increasing bidding strategy for every position k in equilibrium of a K-dimensional assortative position auction, then the equilibrium allocation is always efficient. Let $\beta(x) = \big(\beta_1(x), \cdots, \beta_K(x)\big)$ be the symmetric equilibrium bidding strategy. Since $\beta_k(x)$ is strictly increasing for every k, the bidder with the highest signal will submit the highest bid for position 1 and win position 1. The bidder with the second highest signal will submit the highest bid among the rest of bidders and win position 2, etc. The equilibrium allocation will rank bidders according to their signals and therefore is efficient.

I will next show that an equilibrium of a K-dimensional assortative position auction is efficient only if every bidder uses a symmetric and strictly increasing bidding strategy $\beta_k(x)$ for any position k. Suppose an efficient equilibrium exists in a K-dimensional assortative auction, then a bidder who receives a signal x_i must be placed above a bidder who receives a lower signal $x_j < x_i$ if both bidders receive some position in equilibrium. Pick an arbitrary bidder i, for any position $k \in [1, K]$, take any value $x_i^{'} > x_i$, then there is positive probability that there are exactly $(k-1)$ bidders who receive signals above $x_i^{'}$ and one bidder $j \neq i$ who receives signal $x_j \in (x_i, x_i^{'})$. Efficiency requires that bidder i wins position k if bidder i receives signal $x_i^{'}$, and bidder j wins position k if bidder i receives signal x_i. With the K-dimensional assortative ranking rule, bidder i's bid for position k must always be higher than bidder j's bid for position k when receiving $x_i^{'}$, and bidder i's bid for position k must always be lower than bidder

j's bid for position k when receiving x_i:

$$\begin{aligned} \beta_{ik}(x_i^{'}) &> \beta_{jk}(x_j) \\ \beta_{ik}(x_i) &< \beta_{jk}(x_j) \end{aligned} \tag{66}$$

This is only possible when $\beta_{ik}(x_i^{'}) > \beta_{ik}(x_i)$. Therefore, for every bidder i and every position k, we must have

$$x_i^{'} > x_i \quad \rightarrow \quad \beta_{ik}(x_i^{'}) > \beta_{ik}(x_i) \tag{67}$$

which means $\beta_{ik}(x_i)$ is strictly increasing in x_i for every i and every k.

Next, I will show that any efficient equilibrium in a K-dimensional assortative position auction must be symmetric across bidders. Suppose the equilibrium is not symmetric, i.e., there exists some $k \in [1, K]$ and $i \neq j$ s.t. $\beta_{ik}(\hat{x}) \neq \beta_{jk}(\hat{x})$ for some $\hat{x} \in [0, \bar{x}]$. Without loss of generality, assume $\beta_{ik}(\hat{x}) > \beta_{jk}(\hat{x})$. Since $\beta_{ik}(.)$ and $\beta_{jk}(.)$ are continuous, there exists x_i, x_j s.t. $x_i < \hat{x} < x_j$, and $\beta_{ik}(x_i) > \beta_{jk}(x_j)$. There is positive probability that there are exactly $(k-1)$ bidders other than i and j receive signals above x_j. Since $x_i < \hat{x} < x_j$, efficiency requires that bidder j wins position k. However, with the K-dimensional assortative ranking rule, $\beta_{ik}(x_i) > \beta_{jk}(x_j)$ implies that bidder j cannot win position k, which yields a contradiction. Therefore, it is impossible to have $\beta_{ik}(\hat{x}) \neq \beta_{jk}(\hat{x})$ for any i, j, any $k \in [1, K]$, and any value of $\hat{x}$. In any efficient equilibrium, each bidder must use a symmetric bidding strategy $\beta_{ik}(.) = \beta_k(.)$ for every position k. □

Proof of Proposition 3:

Proof. For any arbitrary bidder i, let $g_i^{\{k\}}(y_k, \cdots, y_1|x_i)$ be the joint density of $(Y_k, Y_{k-1}, \cdots, Y_1)$ conditional on $X = x_i$, according to the joint distribution of signals $F(x_1, \cdots, x_N)$. Define $v^{\{k\}}(x_i; y_1, y_2, \cdots, y_k)$ as bidder i's expected value per click conditional on her own signal X equals to x_i, the highest signal Y_1, the second highest signal Y_2, ..., the

k-th highest signal Y_k received by her opponents equals to $(y_1, y_2, \cdots, y_k)$:

$$v^{\{k\}}(x_i; y_1, y_2, \cdots, y_k) = E\Big[v_i \Big| X = x_i, Y_1 = y_1, Y_2 = y_2, \cdots, Y_k = y_k\Big] \tag{68}$$

Suppose all of bidder i's opposing bidders follow the monotonic Bayesian equilibrium bidding strategy $\beta(.) = (\beta_1(.), \beta_2(.), \cdots, \beta_K(.))$ in the K-dimensional GSP auction. Let $\beta_k^{-1}(.)$ denote the inverse function of $\beta_k(.)$. Then bidder i's best response bid $(b_i^{1*}, b_i^{2*}, \cdots, b_i^{K*})$ maximizes

$$\begin{aligned}
&\int_0^{\beta_1^{-1}(b_i^1)} \alpha_1 \Big[v^{\{1\}}(x_i, y_1) - \beta(y_1)\Big] g_i^{\{1\}}(y_1|x_i) dy_1 \\
&+ \int_{\beta_1^{-1}(b_i^1)}^1 \int_0^{\beta_2^{-1}(b_i^2)} \alpha_2 \Big[v^{\{2\}}(x_i, y_1, y_2) - \beta(y_2)\Big] g_i^{\{2\}}(y_2, y_1|x_i) dy_2 dy_1 \\
&+ \int_{\beta_1^{-1}(b_i^1)}^1 \int_{\beta_2^{-1}(b_i^2)}^{y_1} \int_0^{\beta_3^{-1}(b_i^3)} \alpha_3 \Big[v^{\{3\}}(x_i, y_1, y_2, y_3) - \beta(y_3)\Big] g_i^{\{3\}}(y_3, y_2, y_1|x_i) dy_3 dy_2 dy_1 \\
&+ \cdots \\
&+ \int_{\beta_1^{-1}(b_i^1)}^1 \int_{\beta_2^{-1}(b_i^2)}^{y_1} \int_{\beta_3^{-1}(b_i^3)}^{y_2} \cdots \int_0^{\beta_K^{-1}(b_i^K)} \alpha_K \Big[v^{\{K\}}(x_i, y_1, \cdots, y_K) - \beta(y_K)\Big] g_i^{\{K\}}(y_K, \cdots, y_1|x_i) dy_K \cdots dy_1
\end{aligned} \tag{69}$$

Define $\Pi_k^G\big(x_i, y_1, \cdots, y_k\big)$ as

$$\Pi_k^G\big(x_i, y_1, \cdots, y_k\big) = \alpha_k \big[v^{\{k\}}(x_i, y_1, \cdots, y_k) - \beta(y_k)\big] \times g_i^{\{k\}}(y_k, \cdots, y_1|x_i) \tag{70}$$

Then the objective function (69) can be rewritten as

$$
\begin{aligned}
\Pi(b_i|x_i) = & \underbrace{\int_0^{\beta_1^{-1}(b_i^1)} \Pi_1^G\big(x_i, y_1\big) dy_1}_{A_1} \\
& + \underbrace{\int_{\beta_1^{-1}(b_i^1)}^1 \int_0^{\beta_2^{-1}(b_i^2)} \Pi_2^G\big(x_i, y_1, y_2\big) dy_2 dy_1}_{A_2} \\
& + \underbrace{\int_{\beta_1^{-1}(b_i^1)}^1 \int_{\beta_2^{-1}(b_i^2)}^{y_1} \int_0^{\beta_3^{-1}(b_i^3)} \Pi_3^G\big(x_i, y_1, \cdots, y_3\big) dy_3 dy_2 dy_1}_{A_3} \\
& + \cdots \\
& + \underbrace{\int_{\beta_1^{-1}(b_i^1)}^1 \int_{\beta_2^{-1}(b_i^2)}^{y_1} \int_{\beta_3^{-1}(b_i^3)}^{y_2} \cdots \int_0^{\beta_K^{-1}(b_i^K)} \Pi_K^G\big(x_i, y_1, \cdots, y_K\big) dy_K \cdots dy_1}_{A_K}
\end{aligned}
\tag{71}
$$

Let A_k denote the k-th term in the objective function (71). The definitions of A_1, A_2 and A_3 are shown in the objective function (71). For any $k \geq 3$, A_k is given by

$$
A_k = \underbrace{\int_{\beta_1^{-1}(b_i^1)}^1 \int_{\beta_2^{-1}(b_i^2)}^{y_1} \cdots \int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}}}_{(k-1)} \int_0^{\beta_k^{-1}(b_i^k)} \Pi_k^G\big(x_i, y_1, \cdots, y_k\big) dy_k \cdots dy_2 dy_1 \tag{72}
$$

The first order condition with respect to $b_i^1, b_i^2, \cdots, b_i^K$ is given by

$$
\begin{aligned}
\frac{\partial A_1}{\partial b_i^1} + \frac{\partial A_2}{\partial b_i^1} + \frac{\partial A_3}{\partial b_i^1} + \cdots + \frac{\partial A_K}{\partial b_i^1} &= 0 \\
\frac{\partial A_2}{\partial b_i^2} + \frac{\partial A_3}{\partial b_i^2} + \cdots + \frac{\partial A_K}{\partial b_i^2} &= 0 \\
\cdots \\
\frac{\partial A_{K-1}}{\partial b_i^{K-1}} + \frac{\partial A_K}{\partial b_i^{K-1}} &= 0 \\
\frac{\partial A_K}{\partial b_i^K} &= 0
\end{aligned}
\tag{73}
$$

Since each b_i^k enters $A_k, A_{k+1}, \cdots, A_K$, but does not enter any $A_{k'}$ with $k' < k$.

For any $1 \leq k \leq K$, take derivative of A_k with respect to b_i^k and replacing b_i^{n*} by $\beta_n(x_i)$ for all $n \in \{1, 2, \cdots, K\}$ yields

$$\begin{aligned}
\frac{\partial A_k}{\partial b_i^k} =& \frac{1}{\beta_k'(\beta_k^{-1}(b_i^k))} \int_{\beta_1^{-1}(b_i^1)}^{1} \cdots \int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}} \Pi_k^G\Big(x_i, y_1, \cdots, y_{k-1}, \beta_k^{-1}(b_i^k)\Big) dy_{k-1} \cdots dy_1 \\
=& \frac{1}{\beta_k'(x_i)} \int_{x_i}^{1} \cdots \int_{x_i}^{y_{k-2}} \Pi_k^G\Big(x_i, y_1, \cdots, y_{k-1}, x_i\Big) dy_{k-1} \cdots dy_1 \\
=& \frac{g_k(x_i|x_i)}{\beta_k'(x_i)} \alpha_k \Big[v^k(x_i, x_i) - b_i^k\Big]
\end{aligned} \tag{74}$$

Take derivative of A_{k+1} with respect to b_i^k, and replace b_i^n by $\beta_n(x_i)$ for all $n \in \{1, 2, \cdots, K\}$ yields

$$\begin{aligned}
\frac{\partial A_{k+1}}{\partial b_i^k} =& -\frac{1}{\beta_k'(\beta_k^{-1}(b_i^k))} \int_{\beta_1^{-1}(b_i^1)}^{1} \cdots \int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}} \int_0^{\beta_{k+1}^{-1}(b_i^{k+1})} \Pi_{k+1}^G\Big(x_i, y_1, \cdots, \beta_k^{-1}(b_i^k), y_{k+1}\Big) dy_{k+1} dy_{k-1} \cdots dy_1 \\
=& -\frac{1}{\beta_k'(x_i)} \int_{x_i}^{1} \cdots \int_{x_i}^{y_{k-2}} \int_0^{x_i} \Pi_{k+1}^G\Big(x_i, y_1, \cdots, x_i, y_{k+1}\Big) dy_{k+1} dy_{k-1} \cdots dy_1 \\
=& -\frac{g_k(x_i|x_i)}{\beta_k'(x_i)} \alpha_k \Big[v^k(x_i, x_i) - \int_0^{x_i} \beta_{k+1}(y_{k+1}) dG_{k+1}\big(y_{k+1}\big|x_i, x_i\big)\Big]
\end{aligned} \tag{75}$$

Take derivative of A_{k+2} with respect to b_i^k, and replace b_i^n by $\beta_n(x_i)$ for all $n \in \{1, 2, \cdots, K\}$ yields

$$\begin{aligned}
&\frac{\partial A_{k+2}}{\partial b_i^k} = \frac{1}{\beta_k'(\beta_k^{-1}(b_i^k))} \times \\
&\int_{\beta_1^{-1}(b_i^1)}^{1} \cdots \int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}} \int_{\beta_{k+1}^{-1}(b_i^{k+1})}^{\beta_k^{-1}(b_i^k)} \int_0^{\beta_{k+2}^{-1}(b_i^{k+2})} \Pi_{k+2}^G\Big(x_i, y_1, \cdots, \beta_k^{-1}(b_i^k), y_{k+1}, y_{k+2}\Big) dy_{k+2} dy_{k+1} dy_{k-1} \cdots dy_1 \\
=& -\frac{1}{\beta_k'(x_i)} \int_{x_i}^{1} \cdots \int_{x_i}^{y_{k-2}} \int_{x_i}^{x_i} \int_0^{x_i} \Pi_{k+2}^G\Big(x_i, y_1, \cdots, x_i, y_{k+1}, y_{k+2}\Big) dy_{k+2} dy_{k+1} dy_{k-1} \cdots dy_1 \\
=& 0
\end{aligned} \tag{76}$$

This is because the integral of any continuous function on $[x_i, x_i]$ is zero. For any A_n with $n \geq k+2$, $\frac{\partial A_n}{\partial b_i^k}$ also contains an integral on $[x_i, x_i]$. Therefore,

$$\frac{\partial A_n}{\partial b_i^k} = 0, \quad \forall\, n \neq k, k+1 \tag{77}$$

Therefore, the K first order conditions of the objective function characterized in (73) can be simplified to

$$\begin{aligned} \frac{\partial A_k}{\partial b_i^k} + \frac{\partial A_{k+1}}{\partial b_i^k} &= 0, \quad \forall k \in \{1, 2, \cdots, K-1\} \\ \frac{\partial A_K}{\partial b_i^K} &= 0 \end{aligned} \tag{78}$$

For the last position K, the equilibrium bid $b_i^{K*} = \beta_K(x_i)$ is defined by $\frac{\partial A_K}{\partial b_i^K} = 0$, i.e.,

$$\frac{g_K(x_i|x_i)}{\beta_K'(x_i)} \alpha_K \big[v^K(x_i, x_i) - \beta_K(x_i)\big] = 0 \tag{79}$$

so the Bayesian equilibrium bidding strategy for the last position K in the K-dimensional GSP auction is

$$\beta_K(x_i) = v^K(x_i, x_i) \tag{80}$$

For any position $k \in \{1, 2, \cdots, K-1\}$, the equilibrium bid $b_i^{k*} = \beta_k(x_i)$ is characterized by $\frac{\partial A_k}{\partial b_i^k} + \frac{\partial A_{k+1}}{\partial b_i^k} = 0$, i.e.,

$$\frac{g_k(x_i|x_i)}{\beta_k'(x_i)} \Big[\alpha_k \big[v^k(x_i, x_i) - \beta_k(x_i)\big] - \alpha_{k+1} \big[v^k(x_i, x_i) - \int_0^{x_i} \beta_{k+1}(y_{k+1}) dG_{k+1|k}\big(y_k|x_i, x_i\big)\big]\Big] = 0 \tag{81}$$

Rearranging equation (81) gives the equilibrium bidding strategy $\beta_k(x_i)$ for any position above K in the K-dimensional GSP auction:

$$\beta_k(x_i) = v^k(x_i, x_i) - \frac{\alpha_{k+1}}{\alpha_k} \Big[v^k(x_i, x_i) - \int_0^{x_i} \beta_{k+1}(y_{k+1}) dG_{k+1|k}\big(y_k|x_i, x_i\big)\Big], \quad \forall k \in [1, K-1] \tag{82}$$

□

Proof of Proposition 4:

Proof. Suppose all of bidder i's opposing bidders follow a monotonic Bayesian equilibrium bidding strategy $\beta(x) = \big(\beta_1(.), \beta_2(.), \cdots, \beta_K(.)\big)$ in the K-dimensional VCG auction. Let $\beta_k^{-1}(.)$ denote the inverse function of $\beta_k(.)$. Let $g_i^{\{K\}}(y_K, \cdots, y_1|x_i)$ be the joint density of $(Y_K, Y_{K-1}, \cdots, Y_1)$ conditional on $X = x_i$. Let $v^{\{K\}}(x_i; y_1, y_2, \cdots, y_K)$ be bidder i's expected value per click conditional on her own signal X equals to x_i, the highest signal Y_1, the second highest signal Y_2, ... the K-th highest signal Y_K received by her opponents equal to $(y_1, y_2, \cdots, y_K)$. Define $\Pi_k^V\big(x_i, y_1, \cdots, y_K\big)$ as

$$\Pi_k^V\big(x_i, y_1, \cdots, y_K\big) = \Big[\alpha_k v^{\{K\}}(x_i, y_1, \cdots, y_K) - \sum_{j=k}^{K}(\alpha_j - \alpha_{j+1})\beta_j(y_j)\Big] \times g_i^{\{K\}}(y_K, \cdots, y_1|x_i) \tag{83}$$

Then bidder i's best response bid $(b_i^{1*}, b_i^{2*}, \cdots, b_i^{K*})$ maximizes

$$\begin{aligned}
\Pi(b_i|x_i) = & \underbrace{\int_0^{\beta_1^{-1}(b_i^1)} \int_0^{y_1} \int_0^{y_2} \cdots \int_0^{y_{K-1}} \Pi_1^V\big(x_i, y_1, y_2, \cdots, y_K\big) dy_K \cdots dy_1}_{B_1} \\
& + \underbrace{\int_{\beta_1^{-1}(b_i^1)}^{1} \int_0^{\beta_2^{-1}(b_i^2)} \int_0^{y_2} \cdots \int_0^{y_{K-1}} \Pi_2^V\big(x_i, y_1, y_2, \cdots, y_K\big) dy_K \cdots dy_1}_{B_2} \\
& + \underbrace{\int_{\beta_1^{-1}(b_i^1)}^{1} \int_{\beta_2^{-1}(b_i^2)}^{y_1} \int_0^{\beta_3^{-1}(b_i^3)} \cdots \int_0^{y_{K-1}} \Pi_3^V\big(x_i, y_1, y_2, \cdots, y_K\big) dy_K \cdots dy_1}_{B_3} \\
& + \cdots \\
& + \underbrace{\int_{\beta_1^{-1}(b_i^1)}^{1} \int_{\beta_2^{-1}(b_i^2)}^{y_1} \int_{\beta_3^{-1}(b_i^3)}^{y_2} \cdots \int_0^{\beta_K^{-1}(b_i^K)} \Pi_K^V\big(x_i, y_1, y_2, \cdots, y_K\big) dy_K \cdots dy_1}_{B_K}
\end{aligned} \tag{84}$$

Let B_k denote the k-th term in equation (84). B_1, B_2 and B_3 are given in equation (84). For all $k \geq 3$, each B_k can be expressed as

$$B_k = \underbrace{\int_{\beta_1^{-1}(b_i^1)}^{1} \int_{\beta_2^{-1}(b_i^2)}^{y_1} \cdots \int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}}}_{(k-1)} \int_0^{\beta_k^{-1}(b_i^k)} \underbrace{\int_0^{y_k} \cdots \int_0^{y_{K-1}}}_{(K-k)} \Pi_k^V\big(x_i, y_1, y_2, \cdots, y_K\big) dy_K \cdots dy_1 \tag{85}$$

Since each b_i^k only enters $B_k, B_{k+1}, \cdots, B_K$, but not enter any $B_{k'}$ with $k' < k$, the first order condition of the objective function (84) with respect to $(b_i^1, b_i^2, \cdots, b_i^K)$ is given by

$$\begin{aligned}
\frac{\partial B_1}{\partial b_i^1}+\frac{\partial B_2}{\partial b_i^1}+\frac{\partial B_3}{\partial b_i^1}+\cdots+\frac{\partial B_K}{\partial b_i^1}&=0\\
\frac{\partial B_2}{\partial b_i^2}+\frac{\partial B_3}{\partial b_i^2}+\cdots+\frac{\partial B_K}{\partial b_i^2}&=0\\
\cdots\\
\frac{\partial B_{K-1}}{\partial b_i^{K-1}}+\frac{\partial B_K}{\partial b_i^{K-1}}&=0\\
\frac{\partial B_K}{\partial b_i^K}&=0
\end{aligned} \tag{86}$$

Take derivative of B_k with respect to b_i^k, and replace b_i^n by $\beta_n(x_i)$ for all $n \in \{1,2,\cdots,K\}$ yields

$$\begin{aligned}
\frac{dB_k}{db_i^k}=&\frac{1}{\beta_k'(\beta_k^{-1}(b_i^k))}\times\\
&\int_{\beta_1^{-1}(b_i^1)}^1\cdots\int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}}\int_0^{\beta_k^{-1}(b_i^k)}\cdots\int_0^{y_{K-1}}\Pi_k^V\Big(x_i,y_1,\cdots,y_{k-1},\beta_k^{-1}(b_i^k),y_{k+1},\cdots,y_K\Big)dy_K\cdots dy_1\\
=&\frac{1}{\beta_k'(x_i)}\int_{x_i}^1\cdots\int_{x_i}^{y_{k-2}}\int_0^{x_i}\cdots\int_0^{y_{K-1}}\Pi_k^V\Big(x_i,y_1,\cdots,y_{k-1},x_i,y_{k+1},\cdots,y_K\Big)dy_K\cdots dy_1
\end{aligned} \tag{87}$$

Take derivative of B_{k+1} with respect to b_i^k, and replacing b_i^n by $\beta_n(x_i)$ for all $n \in \{1,2,\cdots,K\}$ yields

$$\begin{aligned}
\frac{dB_{k+1}}{db_i^k}=&-\frac{1}{\beta_k'(\beta_k^{-1}(b_i^k))}\times\\
&\int_{\beta_1^{-1}(b_i^1)}^1\cdots\int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}}\int_0^{\beta_{k+1}^{-1}(b_i^{k+1})}\cdots\int_0^{y_{K-1}}\Pi_{k+1}^V\Big(x_i,y_1,\cdots,y_{k-1},\beta_k^{-1}(b_i^k),y_{k+1},\cdots,y_K\Big)dy_K\cdots dy_1\\
=&-\frac{1}{\beta_k'(x_i)}\int_{x_i}^1\cdots\int_{x_i}^{y_{k-2}}\int_0^{x_i}\cdots\int_0^{y_{K-1}}\Pi_{k+1}^V\Big(x_i,y_1,\cdots,y_{k-1},x_i,y_{k+1},\cdots,y_K\Big)dy_K\cdots dy_1
\end{aligned} \tag{88}$$

Take derivative of B_{k+2} with respect to b_i^k, and replace b_i^n by $\beta_n(x_i)$ for all $n \in$

$\{1, 2, \cdots, K\}$ yields

$$\begin{aligned}\frac{dB_{k+2}}{db_i^k} =& -\frac{1}{\beta_k^{'}(\beta_k^{-1}(b_i^k))} \times \\ & \int_{\beta_1^{-1}(b_i^1)}^{1} \cdots \int_{\beta_{k-1}^{-1}(b_i^{k-1})}^{y_{k-2}} \int_{\beta_{k+1}^{-1}(b_i^{k+1})}^{\beta_k^{-1}(b_i^k)} \int_{0}^{\beta_{k+2}^{-1}(b_i^{k+2})} \cdots \int_0^{y_{K-1}} \Pi_{k+2}^V\Big(x_i, y_1, \cdots, \beta_k^{-1}(b_i^k), \cdots, y_K\Big) dy_K \cdots dy_1 \\ =& -\frac{1}{\beta_k^{'}(x_i)} \int_{x_i}^{1} \cdots \int_{x_i}^{y_{k-2}} \int_{x_i}^{x_i} \int_0^{x_i} \cdots \int_0^{y_{K-1}} \Pi_{k+2}^V\Big(x_i, y_1, \cdots, y_{k-1}, x_i, y_{k+1}, \cdots, y_K\Big) dy_K \cdots dy_1 \\ =& 0\end{aligned} \tag{89}$$

since the integral of any continuous function on $[x_i, x_i]$ is zero. At the equilibrium where $b_i = \beta(x_i)$, $\frac{dB_n}{db_i^k}$ contains an integral on $[x_i, x_i]$ for any B_n with $n \geq k+2$, so $\frac{dB_n}{db_i^k} = 0$ for all $n \neq k, k+1$. Therefore, the first order conditions characterized in equation (86) becomes

$$\begin{aligned}&\frac{dB_k}{db_i^k} + \frac{dB_{k+1}}{db_i^k} = 0, \quad \forall k \in [1, K-1] \\ &\frac{dB_K}{db_i^K} = 0\end{aligned} \tag{90}$$

For the last position K, the equilibrium bid $b_i^{K*} = \beta_K(x_i)$ is characterized by $\frac{dB_K}{db_i^K} = 0$:

$$\begin{aligned}&\frac{1}{\beta_K^{'}(x_i)} \int_{x_i}^{1} \cdots \int_{x_i}^{y_{K-2}} \int_0^{x_i} \Pi_K^V\Big(x_i, y_1, \cdots, y_{k-1}, x_i\Big) dy_{K-1} \cdots dy_1 \\ =& \frac{1}{\beta_K^{'}(x_i)} \int_{x_i}^{1} \cdots \int_{x_i}^{y_{K-2}} \int_0^{x_i} \alpha_K \Big[v^{\{K\}}(x_i, y_1, \cdots, y_{K-1}, x_i) - \beta_K(x_i)\Big] g_i^{\{K\}}(x_i, y_{K-1}, \cdots, y_1|x_i) dy_K \cdots dy_1 \\ =& \frac{g_K(x_i|x_i)}{\beta_K^{'}(x_i)} \alpha_K \Big[v^K(x_i, x_i) - \beta_K(x_i)\Big] \\ =& 0\end{aligned} \tag{91}$$

so the equilibrium bidding strategy for the last position K in the K-dimensional VCG auction is given by

$$\beta_K(x_i) = v^K(x_i, x_i) \tag{92}$$

For any position $1 \leq k \leq K-1$, the equilibrium bid $b_i^{k*} = \beta_k(x_i)$ is characterized by $\frac{dB_k}{db_i^k} + \frac{dB_{k+1}}{db_i^k} = 0$:

$$
\begin{aligned}
&\frac{1}{\beta_k'(x_i)} \int_{x_i}^{1} \cdots \int_{0}^{x_i} \cdots \int_{0}^{y_{K-1}} \Big[\Pi_k^V\big(x_i, y_1, \cdots, x_i, \cdots, y_K\big) - \Pi_{k+1}^V\big(x_i, y_1, \cdots, x_i, \cdots, y_K\big)\Big] dy_K \cdots dy_1 \\
=&\frac{1}{\beta_k'(x_i)} \int_{x_i}^{1} \cdots \int_{0}^{x_i} \cdots \int_{0}^{y_{K-1}} (\alpha_k - \alpha_{k+1}) \Big[v^{\{K\}}(x_i, y_1, \cdots, x_i, \cdots, y_K) - \beta_k(x_i)\Big] dy_K \cdots dy_1 \\
=&\frac{g_k(x_i|x_i)}{\beta_k'(x_i)} (\alpha_k - \alpha_{k+1}) \Big[v^k(x_i, x_i) - \beta_k(x_i)\Big] \\
=&0
\end{aligned}
\tag{93}
$$

Therefore, for any position above the last position K, the equilibrium bidding strategy $\beta_k(x_i)$ in the K-dimensional VCG auction is given by

$$
\beta_k(x_i) = v^k(x_i, x_i), \quad \forall k \in \{1, 2, \cdots, K-1\} \tag{94}
$$

□

Proof of Proposition 5:

Proof. First consider the case when no bidder has dropped out. When there are more than K bidders remaining in the auction, each bidder will not drop out until the expected payoff from the last position K falls below zero. Suppose all the opposing bidders adopt strategy b_N^* defined in proposition 5, $b_N^*(x_i) = v^{(K)}(x_i, x_i, \cdots, x_i)$. When all bidders are in the auction, at any price p, bidder i wins the last position K by dropping out right now only if there are $(N-K)$ bidders drop out simultaneously at this price, i.e., the lowest $(N-K)$ value bidders have the same signal $Y_K = Y_{K+1} = \cdots = Y_{N-1} = y_K$. Therefore, given that all opponents follow strategy $b_N^*(x)$, bidder i's expected value conditional on winning K is

$$
\alpha_K v^{(K)}\big(x_i, y_K, \cdots, y_K\big) = \alpha_K E[v_i | X = x_i, Y_K = y_K, Y_{K+1} = y_K, \cdots, Y_{N-1} = y_K] \tag{95}
$$

Bidder i's expected payment conditional on winning K is

$$\alpha_K v^{(K)}\big(y_K, y_K, \cdots, y_K\big) = \alpha_K E[v_i|X = y_K, Y_K = y_K, Y_{K+1} = y_K, \cdots, Y_{N-1} = y_K] \tag{96}$$

The expected payoff from the last position K is non-negative for bidder i if and only if $x_i \geq y_K$. By using strategy b_N^*, bidder i will win position K or some position above K if and only if $x_i \geq y_K$, so b_N^* is the best response bidding strategy for each bidder i when all bidders are sill in the auction, assuming all other bidders also adopt strategy b_N^*. This is an ex-post equilibrium, since b_N^* is bidder i's optimal strategy for any realization of opposing bidders' signals x_{-i}.

Next, consider the case when $(N-n)$ bidders have dropped out, but $n \geq K+1$ bidders are still in the auction so that the allocation of no position has been determined. Similar to the case with N active bidders, each bidder will not drop out until the expected payoff from the last position K falls below zero. However, the expected payoff from the last position is now calculated conditional on the revealed signals of the $(N-n)$ drop-out bidders, $Y_n = y_n, \cdots, Y_{N-1} = y_{N-1}$, in which $y_n, y_{n+1}, \cdots, y_{N-2}, y_{N-1}$ are inferred from $b_N^*(y_{N-1}) = p_N$, $b_{N-1}^*(y_{N-2}|p_N) = p_{N-1}$, $b_{n+1}^*(y_n|p_N, \cdots, p_{n+2}) = p_{n+1}$. Assume all the remaining opposing bidders adopt strategy b_n^*. At any price p, bidder i will win the position K by dropping out at the current price only if the lowest-value $(n-K)$ bidders among the active bidders drop out simultaneously, i.e., they have the same signal $Y_K = \cdots = Y_{n-1} = y_K$. Bidder i's expected value upon winning K is

$$\begin{aligned}&\alpha_K v^{(K)}\big(x_i, \underbrace{y_K, \cdots, y_K}_{(n-K)}, \underbrace{y_n, y_{n+1}, \cdots, y_{N-1}}_{(N-n)\text{ lowest signals}}\big) \\ &= \alpha_K E\big[v_i \big| X = x_i, Y_K = y_K, Y_{K+1} = y_K, \cdots, Y_{n-1} = y_K, Y_n = y_n, \cdots, Y_{N-1} = y_{N-1}\big]\end{aligned} \tag{97}$$

Her payment upon winning K is

$$\begin{aligned}&\alpha_K v^{(K)}\big(y_K, \underbrace{y_K, \cdots, y_K}_{(n-K)}, \underbrace{y_n, y_{n+1}, \cdots, y_{N-1}}_{(N-n)\text{ lowest signals}}\big) \\ &= \alpha_K E\big[v_i \big| X = y_K, Y_K = y_K, Y_{K+1} = y_K, \cdots, Y_{n-1} = y_K, Y_n = y_n, \cdots, Y_{N-1} = y_{N-1}\big]\end{aligned} \tag{98}$$

Therefore, it is profitable to stay in the auction if and only if $x_i \geq y_K$. By using bidding strategy b_n^*, bidder i will win a position no lower than K if and only if $x_i \geq y_K$, so b_n^* is the best response bidding strategy for each bidder i when there are $K < n < N$ bidders in the auction. This is an ex-post equilibrium, since b_n^* is the best response given any realization of other bidders' signals.

Next, consider the case when $n \leq K$ bidders are left in the auction. When there are $n \leq K$ bidders left in the auction, all the remaining bidders will win some position, so the drop-out price of each bidder i only affect which position she gets. In equilibrium, a bidder with signal x_i should be indifferent between getting the current lowest position n at price p_{n+1} and the next best position $(n-1)$ at a higher price. Note that bidder i wins position $(n-1)$ at a higher price b only if the lowest-value remaining bidder drops out at b. Assuming that all remaining opposing bidders adopt strategy b_n^*, bidder i's expected payoff from winning the next best position $(n-1)$ given the revealed signals $(y_{n-1}, \cdots, y_N)$ is

$$\begin{aligned} E\Pi_{n-1} =& \alpha_{n-1}\Big[v^{(n-1)}\big(x_i, y_{n-1}, y_n, \cdots, y_N\big) - b\Big] \\ =& \alpha_{n-1}\Big[v^{(n-1)}\big(x_i, y_{n-1}, y_n, \cdots, y_N\big) - v^{(n-1)}\big(y_{n-1}, y_{n-1}, y_n, \cdots, y_N\big)\Big] \\ & + \alpha_n\Big[v^{(n-1)}\big(y_{n-1}, y_{n-1}, y_n, \cdots, y_N\big) - p_{n+1}\Big] \end{aligned} \tag{99}$$

since

$$b = v^{(n-1)}\big(y_{n-1}, y_{n-1}, y_n, \cdots, y_N\big) - \frac{\alpha_n}{\alpha_{n-1}}\Big[v^{(n-1)}\big(y_{n-1}, y_{n-1}, y_n, \cdots, y_N\big) - p_{n+1}\Big] \tag{100}$$

Bidder i's expected payoff from winning the current lowest position n, given $(Y_{n-1}, \cdots, Y_N)$ is

$$E\Pi_n = \alpha_n\Big[v^{(n-1)}(x_i, y_{n-1}, y_n \cdots, y_N) - p_{n+1}\Big] \tag{101}$$

Subtracting equation (101) from equation (99), the expected payoff from staying in the auction and getting position $(n-1)$ is higher than the expected payoff from dropping out right now and getting position n if and only if

$$E\Pi_{n-1} - E\Pi_n = (\alpha_{n-1} - \alpha_n)\Big[v^{(n-1)}(x_i, y_{n-1}, y_n \cdots, y_N) - v^{(n-1)}(y_{n-1}, y_{n-1}, y_n \cdots, y_N)\Big] \geq 0 \tag{102}$$

Inequality (102) holds if and only if $x_i \geq y_{n-1}$. Therefore, by using bidding strategy b_n^*, bidder i wins a position no lower than $(n-1)$ if and only if $x_i \geq y_{n-1}$, so b_n^* is the best response bidding strategy for bidder i when there are $n < K$ bidders remain in the auction. This is an ex-post equilibrium at the time when n bidders are left in the auction, since b_n^* is bidder i's optimal strategy for any realization of the other bidders signals x_{-i}. Therefore, $(b^*, \cdots, b^*)$ characterized in Proposition 5 is an ex-post equilibrium in the Generalized English Auction with interdependent values. □

Proof of Proposition 6::

Proof. I first compare expected revenues of the K-dimensional GSP auction and the K-dimensional VCG auction, and then compare expected revenues of the K-dimensional VCG auction and the GEA.

Let $\beta^V(x_i) = \big(\beta_1^V(x_i), \beta_2^V(x_i), \cdots, \beta_K^V(x_i)\big)$ and $\beta^G(x_i) = \big(\beta_1^G(x_i), \beta_2^G(x_i), \cdots, \beta_K^G(x_i)\big)$ denote the Bayesian equilibrium bidding strategies in the K-dimensional VCG auction and K-dimensional GSP auction, respectively. According to the characterization of $\beta^V(x_i)$ and $\beta^G(x_i)$ in Propositions 3 and 4, the expected prices for the last position K in the K-dimensional VCG auction and the K-dimensional GSP auction are as follows:

$$\begin{aligned} E\Big[p^{V,(K)}\Big] &= \alpha_K E\Big[\beta_K^V(Y_K)\Big|\{Y_{K-1} > X > Y_K\}\Big] = \alpha_K E\Big[v^K(Y_K, Y_K)\Big|\{Y_{K-1} > X > Y_K\}\Big] \\ E\Big[p^{G,(K)}\Big] &= \alpha_K E\Big[\beta_K^G(Y_K)\Big|\{Y_{K-1} > X > Y_K\}\Big] = \alpha_K E\Big[v^K(Y_K, Y_K)\Big|\{Y_{K-1} > X > Y_K\}\Big] \end{aligned} \tag{103}$$

For any position $1 \leq k \leq K-1$, the expected price $E\Big[p^{V,(k)}\Big]$ in the K-dimensional VCG auction and the expected price $E\Big[p^{G,(k)}\Big]$ in the K-dimensional GSP auction are given below:

$$
\begin{aligned}
E\Big[p^{V,(k)}\Big] &= (\alpha_k - \alpha_{k+1})E\Big[\beta_k^V(Y_k)\Big|\{Y_{k-1} > X > Y_k\}\Big] + E\Big[p^{V,(k+1)}\Big] \\
&= (\alpha_k - \alpha_{k+1})E\Big[v^k(Y_k, Y_k)\Big|\{Y_{k-1} > X > Y_k\}\Big] + E\Big[p^{V,(k+1)}\Big] \\
E\Big[p^{G,(k)}\Big] &= \alpha_k E\Big[\beta_k^G(Y_k)\Big|\{Y_{k-1} > X > Y_k\}\Big] \\
&= \alpha_k E\Big[v^k(Y_k, Y_k) - \Big[\frac{\alpha_{k+1}}{\alpha_k}v^k(Y_k, Y_k) - E\big[\beta_{k+1}^G(Y_{k+1})\big]\Big]\Big|\{Y_{k-1} > X > Y_k\}\Big] \\
&= (\alpha_k - \alpha_{k+1})E\Big[v^k(Y_k, Y_k)\Big|\{Y_{k-1} > X > Y_k\}\Big] + E\Big[p^{G,(k+1)}\Big]
\end{aligned}
\tag{104}
$$

Equation (103) and (104) imply that

$$
\begin{aligned}
E\Big[p^{V,(k)}\Big] - E\Big[p^{V,(k+1)}\Big] &= E\Big[p^{G,(k)}\Big] - E\Big[p^{G,(k+1)}\Big], \quad \forall k \in \{1, 2, \cdots, K-1\} \\
E\Big[p^{V,(K)}\Big] &= E\Big[p^{G,(K)}\Big]
\end{aligned}
\tag{105}
$$

which means the expected prices for the last position K are the same, and the expected difference in prices between any two adjacent positions are the same. Therefore,

$$
E\Big[p^{V,(k)}\Big] = E\Big[p^{G,(k)}\Big], \quad \forall k \in \{1, 2, \cdots, K\} \tag{106}
$$

which directly implies that the K-dimensional VCG auction and the K-dimensional GSP auction are revenue equivalent.

Alternatively, the revenue equivalence between the K-dimensional VCG auction and the K-dimensional GSP auction can be proved by showing that the expected payments of each bidder are the same in two auctions. First consider the case of $K = 2$ positions. The expected payments by a bidder with signal x_i in the K-dimensional VCG auction and the K-dimensional GSP auction are given by

$$\begin{aligned} m^V(x_i) =& Pr(x_i \geq Y_1)E\Big[(\alpha_1 - \alpha_2)\underbrace{v^1(Y_1, Y_1)}_{\beta_1^V(Y_1)} + \alpha_2 \underbrace{v^2(Y_2, Y_2)}_{\beta_2^V(Y_2)}\Big| x_i \geq Y_1\Big] \\ &+ Pr(Y_2 \leq x_i < Y_1)E\Big[\alpha_2 \underbrace{v^2(Y_2, Y_2)}_{\beta_2^V(Y_2)}\Big| Y_2 \leq x_i < Y_1\Big] \\ m^G(x_i) =& Pr(x_i \geq Y_1)E\Big[\alpha_1 \underbrace{\Big[v^1(Y_1, Y_1) - \frac{\alpha_2}{\alpha_1}v^1(Y_1, Y_1) + \frac{\alpha_2}{\alpha_1}E[v^2(Y_2, Y_2)|Y_1]\Big]}_{\beta_1^G(Y_1)}\Big| x_i \geq Y_1\Big] \\ &+ Pr(Y_2 \leq x_i < Y_1)E\Big[\alpha_2 \underbrace{v^2(Y_2, Y_2)}_{\beta_2^G(Y_2)}\Big| Y_2 \leq x_i < Y_1\Big] \end{aligned} \tag{107}$$

The only difference between $m^V(x_i)$ and $m^G(x_i)$ comes from the term $E[v^2(Y_2, Y_2)|Y_1 \leq x_i]$ in $m^V(x_i)$ and $E\Big[E[v^2(Y_2, Y_2)|Y_1]\Big|Y_1 \leq x_i\Big]$ in $m^G(x_i)$. According to the Law of Iterated Expectation,

$$E\Big[E[v^2(Y_2, Y_2)|Y_1]\Big|Y_1 \leq x_i\Big] = E\Big[v^2(Y_2, Y_2)\Big|Y_1 \leq x_i\Big] \tag{108}$$

which implies $m^V(x_i) = m^G(x_i)$. Similar argument applies for any $K \geq 2$ positions. Since the expected payments of a bidder with the same signal x_i are the same in two auctions, the K-dimensional GSP auction and the K-dimensional VCG auction are always revenue equivalent.

I next compare expected revenue of the GEA and the K-dimensional VCG auction. The expected prices for the last position K in GEA and K-dimensional VCG auction are as follows:

$$\begin{aligned} E\Big[p^{E,(K)}\Big] &= \alpha_K E\Big[v^{(K)}(Y_K, Y_K; Y_{K+1}, Y_{K+2}, \cdots, Y_{N-1})\Big|\{Y_{K-1} > X > Y_K\}\Big] \\ E\Big[p^{V,(K)}\Big] &= \alpha_K E\Big[v^K(Y_K, Y_K)\Big|\{Y_{K-1} > X > Y_K\}\Big] \end{aligned} \tag{109}$$

According to Milgrom and Weber (1982)'s Linkage Principle, $E[p^{E,(K)}] \geq E[p^{V,(K)}]$. A formal proof is given below:

$$\begin{aligned}
v^K(x_i, y_K) &= E\Big[v_i \Big| X = x_i, Y_K = y_K\Big] \\
&= E\Big[E\big[v_i \big| X, Y_K, Y_{K+1}, \cdots, Y_{N-1}\big] \Big| X = x_i, Y_K = y_K\Big] \\
&= E\Big[v^{(K)}(X, Y_K; Y_{K+1}, \cdots, Y_{N-1}) \Big| X = x_i, Y_K = y_K\Big]
\end{aligned} \tag{110}$$

For $x_i > y_K$, we have

$$\begin{aligned}
v^K(y_K, y_K) &= E\Big[v^{(K)}(X, Y_K; Y_{K+1}, \cdots, Y_{N-1}) \Big| X = y_K, Y_K = y_K\Big] \\
&= E\Big[v^{(K)}(Y_K, Y_K; Y_{K+1}, \cdots, Y_{N-1}) \Big| X = y_K, Y_K = y_K\Big] \\
&\leq E\Big[v^{(K)}(Y_K, Y_K; Y_{K+1}, \cdots, Y_{N-1}) \Big| X = x_i, Y_K = y_K\Big]
\end{aligned} \tag{111}$$

Therefore,

$$\begin{aligned}
E\Big[p^{V,(K)}\Big] &= \alpha_K E\Big[v^K(Y_K, Y_K) \Big| \{Y_{K-1} > X > Y_K\}\Big] \\
&\leq \alpha_K E\Big[E\big[v^{(K)}(Y_K, Y_K; Y_{K+1}, \cdots, Y_{N-1}) \big| X, Y_K\big] \Big| \{Y_{K-1} > X > Y_K\}\Big] \\
&= \alpha_K E\Big[v^{(K)}(Y_K, Y_K; Y_{K+1}, \cdots, Y_{N-1}) \Big| \{Y_{K-1} > X > Y_K\}\Big] \\
&= E\Big[p^{E,(K)}\Big]
\end{aligned} \tag{112}$$

so the expected price for the last position K is weakly higher in the GEA than in the K-dimensional VCG auction.

For any position $k < K$, the increment in expected price between position k and position $k+1$ in GEA and K-dimensional VCG auction are as follows:

$$\begin{aligned}
E\Big[p^{E,(k)} - p^{E,(k+1)}\Big] &= (\alpha_k - \alpha_{k+1}) E\Big[v^{(k)}(Y_k, Y_k; Y_{k+1}, \cdots, Y_{N-1}) \Big| \{Y_{k-1} > X > Y_k\}\Big] \\
E\Big[p^{V,(k)} - p^{V,(k+1)}\Big] &= (\alpha_k - \alpha_{k+1}) E\Big[v^k(Y_k, Y_k) \Big| \{Y_{k-1} > X > Y_k\}\Big]
\end{aligned} \tag{113}$$

Applying the Linkage Principle again, we have

$$E\Big[v^k(Y_k, Y_k) \Big| \{Y_{k-1} > X > Y_k\}\Big] \leq E\Big[v^{(k)}(Y_k, Y_k; Y_{k+1}, \cdots, Y_{N-1}) \Big| \{Y_{k-1} > X > Y_k\}\Big] \tag{114}$$

so the increment in expected price between any two adjacent positions is weakly higher in the GEA than in the K-dimensional VCG:

$$E\Big[p^{E,(k)}\Big] - E\Big[p^{E,(k+1)}\Big] \geq E\Big[p^{V,(k)}\Big] - E\Big[p^{V,(k+1)}\Big], \quad \forall k \in \{1, 2, \cdots, K-1\} \tag{115}$$

Since the expected price for the last position is weakly higher in GEA, and the increment in expected price between any two positions above the last position is also weakly higher in GEA, the expected price for every position is weakly higher in the GEA than in the K-dimensional VCG auction. Therefore, expected revenue in the GEA is weakly higher than expected revenue in the K-dimensional VCG auction. □

Proof of Proposition 7[15]**:**

Proof. The proof of Proposition 7 is based on two lemmas. Lemma 5 provides a characterization of ex-post IC and IR mechanism under affiliated signals. Lemma 6 characterizes the ex-ante expected revenue in any ex-post IC and IR mechanism.

Lemma 5. *For any value function $v_i(x_i, x_{-i})$ satisfying assumptions **A1-A3** and signal distribution $F(x)$ satisfying assumptions **A4-A5**, a mechanism (q, p) is ex-post IC and IR if and only if for all bidder i, for any signal profile (x_i, x_{-i}), $q_i(x_i, x_{-i})$ is weakly increasing in x_i, and the ex-post utility $u_i(x_i, x_{-i})$ satisfies*

$$u_i(x_i, x_{-i}) = u_i(0, x_{-i}) + \int_0^{x_i} \Big[\frac{\partial v_i(s, x_{-i})}{\partial s}\Big] q_i(s, x_{-i}) ds, \quad \forall\ x_{-i} \tag{116}$$

$$u_i(0, x_{-i}) \geq 0, \quad \forall\ x_{-i} \tag{117}$$

Proof. I first show that any ex-post IC and IR mechanism satisfies the characterization in Lemma 5, then show that any mechanism satisfying the conditions in Lemma 5 must be ex-post IC and IR.

[15] The proof of Proposition 7 follows from Myerson (1981)[32], Ulku (2013)[37] and Li (2017)[27].

Suppose (q, p) is an ex-post IC and IR mechanism. According to the definition of ex-post IC, for all bidder i, for any true signal profile (x_i, x_{-i}) and bidder i's reported signal $x_i^{'}$,

$$\begin{aligned} u_i(x_i, x_{-i}) &\geq q_i(x_i^{'}, x_{-i})v_i(x_i, x_{-i}) - p_i(x_i^{'}, x_{-i}) \\ &= u_i(x_i^{'}, x_{-i}) + q_i(x_i^{'}, x_{-i})\Big[v_i(x_i, x_{-i}) - v_i(x_i^{'}, x_{-i})\Big] \end{aligned} \tag{118}$$

which implies

$$\begin{aligned} u_i(x_i, x_{-i}) &\geq u_i(x_i^{'}, x_{-i}) + q_i(x_i^{'}, x_{-i})\Big[v_i(x_i, x_{-i}) - v_i(x_i^{'}, x_{-i})\Big] \\ u_i(x_i^{'}, x_{-i}) &\geq u_i(x_i, x_{-i}) + q_i(x_i, x_{-i})\Big[v_i(x_i^{'}, x_{-i}) - v_i(x_i, x_{-i})\Big] \end{aligned} \tag{119}$$

which can be rewritten as

$$\begin{aligned} q_i(x_i^{'}, x_{-i})\Big[v_i(x_i, x_{-i}) - v_i(x_i^{'}, x_{-i})\Big] &\leq u_i(x_i, x_{-i}) - u_i(x_i^{'}, x_{-i}) \\ q_i(x_i, x_{-i})\Big[v_i(x_i, x_{-i}) - v_i(x_i^{'}, x_{-i})\Big] &\geq u_i(x_i, x_{-i}) - u_i(x_i^{'}, x_{-i}) \end{aligned} \tag{120}$$

Inequality (120) implies that $q_i(x_i, x_{-i})$ is weakly increasing in x_i, and $u_i(x_i, x_{-i})$ has partial derivative

$$\frac{\partial u_i(x_i, x_{-i})}{\partial x_i} = q_i(x_i, x_{-i})\frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \tag{121}$$

integrate both sides, get

$$u_i(x_i, x_{-i}) = \int_0^{x_i} \Big[q_i(s, x_{-i})\frac{\partial v_i(s, x_{-i})}{\partial s}\Big] ds + u_i(0, x_{-i}) \tag{122}$$

Ex-post IR implies $u_i(x_i, x_{-i}) \geq 0$ for all i. Since $q_i(x_i, x_{-i})$ is weakly increasing in x_i and $v_i(x_i, x_{-i})$ is strictly increasing in x_i, equation (122) implies that $u_i(0, x_{-i}) \leq u_i(x_i, x_{-i})$ for all x_i, given any x_{-i}, so $u_i(x_i, x_{-i}) \geq 0$ for all x_i, given any x_{-i}, only if $u_i(0, x_{-i}) \geq 0$ given any x_{-i}. Therefore, any ex-post IC and IR mechanism must satisfy equation (122), $q_i(x_i, x_{-i})$ increasing in x_i, and $u_i(0, x_{-i}) \geq 0$.

I next show that any mechanism (q, p) that satisfies equation (122), $q_i(x_i, x_{-i})$

increasing in x_i, and $u_i(0, x_{-i}) \geq 0$ for any x_{-i} must be ex-post IC and IR.

Since $q_i(x_i, x_{-i})$ is weakly increasing in x_i, $\frac{\partial v_i(s,x_{-i})}{\partial s} > 0$, and $u_i(x_i, x_{-i}) = u_i(0, x_{-i}) + \int_0^{x_i} \left[\frac{\partial v_i(s,x_{-i})}{\partial s}\right] q_i(s, x_{-i}) ds$, it is trivial that $u_i(x_i, x_{-i}) \geq u_i(0, x_{-i})$ for all $x_i \geq 0$, given any x_{-i}, so $u_i(0, x_{-i}) \geq 0$ for all x_{-i} implies ex-post IR.

Suppose $x_i < x_i^{'}$, then

$$\begin{aligned} u_i(x_i^{'}, x_{-i}) &= u_i(x_i, x_{-i}) + \int_{x_i}^{x_i^{'}} \left[q_i(s, x_{-i}) \frac{\partial v_i(s, x_{-i})}{\partial s}\right] ds \\ &\geq u_i(x_i, x_{-i}) + \int_{x_i}^{x_i^{'}} \left[q_i(x_i, x_{-i}) \frac{\partial v_i(s, x_{-i})}{\partial s}\right] ds \\ &= u_i(x_i, x_{-i}) + \Big[q_i(x_i, x_{-i}) \big(v_i(x_i^{'}, x_{-i}) - v_i(x_i, x_{-i})\big)\Big] \end{aligned} \tag{123}$$

This directly implies ex-post IC. □

The next lemma provides a characterization of the seller's expected revenue in any ex-post IC and IR mechanism.

Lemma 6. *In any ex-post IC and IR mechanism, the ex-ante expected revenue is given by*

$$\begin{aligned} ER = &\int_x \sum_i \Big\{q_i(x_i, x_{-i}) \Big\{v_i(x_i, x_{-i}) - \frac{1 - F_i(x_i|x_{-i})}{f_i(x_i|x_{-i})} \times \frac{\partial v_i(x_i, x_{-i})}{\partial x_i}\Big\}\Big\} f(x) dx \\ &- \int_{x_{-i}} \sum_i u_i(0, x_{-i}) f_{-i|0}(x_{-i}|0) dx_{-i} \end{aligned} \tag{124}$$

Proof. Following equation (122) in Lemma 5, the ex-ante expected payoff to bidder i

in any ex-post IC and IR mechanism is given by

$$
\begin{aligned}
&E_x\big[u_i(x_i,x_{-i})\big] \\
=&\int_{x_{-i}} u_i(0,x_{-i})dF(x_{-i}|0)+\int_x\int_0^{x_i} q_i(s,x_{-i})\Big[\frac{\partial v_i(s,x_{-i})}{\partial s}\Big]dsf(x)dx \\
=&\int_{x_{-i}} u_i(0,x_{-i})dF(x_{-i}|0)+\int_{x_{-i}}\int_0^{\bar{x}}\int_0^{x_i} q_i(s,x_{-i})\Big[\frac{\partial v_i(s,x_{-i})}{\partial s}\Big]dsf_i(x_i|x_{-i})dx_if_{-i}(x_{-i})dx_{-i} \\
=&\int_{x_{-i}} u_i(0,x_{-i})dF(x_{-i}|0)+\int_{x_{-i}}\int_0^{\bar{x}}\int_s^{\bar{x}} q_i(s,x_{-i})\Big[\frac{\partial v_i(s,x_{-i})}{\partial s}\Big]f_i(x_i|x_{-i})dx_idsf_{-i}(x_{-i})dx_{-i} \\
=&\int_{x_{-i}} u_i(0,x_{-i})dF(x_{-i}|0)+\int_{x_{-i}}\int_0^{\bar{x}}\Big(1-F_i(s|x_{-i})\Big)q_i(s,x_{-i})\Big[\frac{\partial v_i(s,x_{-i})}{\partial s}\Big]dsf_{-i}(x_{-i})dx_{-i} \\
=&\int_{x_{-i}} u_i(0,x_{-i})dF(x_{-i}|0)+\int_{x_{-i}}\int_0^{\bar{x}}\Big(1-F_i(x_i|x_{-i})\Big)q_i(x_i,x_{-i})\Big[\frac{\partial v_i(x_i,x_{-i})}{\partial x_i}\Big]dx_if_{-i}(x_{-i})dx_{-i} \\
=&\int_{x_{-i}} u_i(0,x_{-i})dF(x_{-i}|0)+\int_x\Big[\frac{1-F_i(x_i|x_{-i})}{f_i(x_i|x_{-i})}q_i(x_i,x_{-i})\frac{\partial v_i(x_i,x_{-i})}{\partial x_i}\Big]f(x)dx
\end{aligned}
\tag{125}
$$

The ex-ante expected total surplus of the auction is given by

$$
TS=\sum_i\int_x v_i(x_i,x_{-i})q_i(x_i,x_{-i})f(x)dx \tag{126}
$$

The ex-ante expected revenue equals to the expected total surplus subtracted by the expected total payoff to all bidders:

$$
\begin{aligned}
ER=&\sum_i\int_x v_i(x_i,x_{-i})q_i(x_i,x_{-i})f(x)dx \\
&-\sum_i\Big\{\int_{x_{-i}} u_i(0,x_{-i})f(x_{-i}|0)dx_{-i}+\int_x\Big\{\frac{1-F_i(x_i|x_{-i})}{f_i(x_i|x_{-i})}q(x_i,x_{-i})\frac{\partial v_i(x_i,x_{-i})}{\partial x_i}\Big\}f(x)dx\Big\} \\
=&\sum_i\int_x\Big\{q_i(x_i,x_{-i})\times\Big\{v_i(x_i,x_{-i})-\frac{1-F_i(x_i|x_{-i})}{f_i(x_i|x_{-i})}\frac{\partial v_i(x_i,x_{-i})}{\partial x_i}\Big\}\Big\}f(x)dx \\
&-\sum_i\int_{x_{-i}} u_i(0,x_{-i})f(x_{-i}|0)dx_{-i}
\end{aligned}
\tag{127}
$$

□

According to the definition of marginal revenue $MR_i(x_i,x_{-i})$, the seller's problem is to maximize

$$ER = \int_x \sum_i \Big\{ q_i(x_i, x_{-i}) MR_i(x_i, x_{-i}) \Big\} f(x) dx - \int_{x_{-i}} \sum_i u_i(0, x_{-i}) f_{-i|0}(x_{-i}|0) dx_{-i} \tag{128}$$

subject to no reserve price, $u_i(0, x_{-i}) \geq 0$ for any x_{-i}, $q_i(x_i, x_{-i})$ increasing in x_i, and the feasibility constraint. When MR_i is strictly increasing in x_i, the expected revenue can be maximized by setting $u_i(0, x_{-i}) = 0$ for all x_{-i}, and allocating higher CTR to bidders with higher MR_i. Therefore, under regularity condition **R2**, the optimal allocation rule q^* is given by

$$q_i^*(x_i, x_{-i}) = \begin{cases} \alpha_k & \text{if} \quad \hat{X}^k(x_{-i}) \leq x_i < \hat{X}^{k-1}(x_{-i}) \\ 0 & \text{if} \quad x_i < \hat{X}^K(x_{-i}) \end{cases} \tag{129}$$

in which $[\hat{X}^k(x_{-i}), \hat{X}^{k-1}(x_{-i})]$ is the interval of value that bidder i's signal can take such that bidder i has the k-th highest $MR_i(x_i, x_{-i})$ given her opponents' report x_{-i}.

The ex-post IC and IR conditions given in Lemma 5 can be jointly written as

$$q_i(x_i, x_{-i}) v_i(x_i, x_{-i}) - \int_0^{x_i} q_i(s, x_{-i}) \frac{\partial v_i(s, x_{-i})}{\partial s} ds - p_i(x_i, x_{-i}) = u_i(0, x_{-i}) \geq 0, \quad \forall x_{-i} \tag{130}$$

for all bidder i. Choose $p_i^*(x_i, x_{-i}) = q_i^*(x_i, x_{-i}) v_i(x_i, x_{-i}) - \int_0^{x_i} q_i^*(s, x_{-i}) \frac{\partial v_i(s, x_{-i})}{\partial s} ds$, then $p_i^*(x_i, x_{-i})$ satisfies both constraint. Therefore, (q^*, p^*) is an optimal auction subject to no reserve price among all ex-post IC and IR mechanisms. □

Proof of Proposition 8[16]**:**

Proof. To show that (q^*, p^*) characterized in Proposition 7 is optimal subject to no reserve price among all Bayesian IC and IR mechanisms when bidders have independent signals, I first characterize the optimal Bayesian mechanism subject to no reserve price with independent signals, then show it is equivalent to (q^*, p^*). The proof is based on

[16] The proof of Proposition 8 follows from Myerson (1981)[32].

two lemmas presented below:

Lemma 7. *For any value function $v_i(x_i, x_{-i})$ satisfying assumptions **A1**-**A3**, when bidders' signals are independently and identically distributed, a mechanism (q, p) is Bayesian IC and IR if for every bidder i, for any report of signals $x = (x_i, x_{-i})$, the expected CTR $q_i(x_i, x_{-i})$ is weakly increasing in x_i, and the interim expected utility $U_i(x_i)$ satisfies*

$$U_i(x_i) = U_i(0) + \int_{x_{-i}} \int_0^{x_i} \Big[\frac{\partial v_i(s, x_{-i})}{\partial s}\Big] q_i(s, x_{-i}) ds f_{-i}(x_{-i}) dx_{-i} \tag{131}$$

$$U_i(0) \geq 0 \tag{132}$$

Proof. I first show that any Bayesian IC and IR mechanism can be characterized by the conditions in Lemma 7, then finish the proof by showing that any mechanism satisfying the characterization in Lemma 7 must be Bayesian IC and IR.

According to the definition of Bayesian IC mechanism, for all bidder i, for any signal profile (x_i, x_{-i}) and bidder i's reported signal $x_i^{'}$,

$$\begin{aligned} U_i(x_i) &\geq \int_{x_{-i}} \Big[q_i(x_i^{'}, x_{-i}) v_i(x_i, x_{-i}) - p_i(x_i^{'}, x_{-i})\Big] f_{-i}(x_{-i}) dx_{-i} \\ &= U_i(x_i^{'}) + \int_{x_{-i}} \Big[q_i(x_i^{'}, x_{-i}) \Big(v_i(x_i, x_{-i}) - v_i(x_i^{'}, x_{-i})\Big)\Big] f_{-i}(x_{-i}) dx_{-i} \end{aligned} \tag{133}$$

which implies

$$\begin{aligned} U_i(x_i) &\geq U_i(x_i^{'}) + \int_{x_{-i}} \Big[q_i(x_i^{'}, x_{-i}) \Big(v_i(x_i, x_{-i}) - v_i(x_i^{'}, x_{-i})\Big)\Big] f_{-i}(x_{-i}) dx_{-i} \\ U_i(x_i^{'}) &\geq U_i(x_i) + \int_{x_{-i}} \Big[q_i(x_i, x_{-i}) \Big(v_i(x_i^{'}, x_{-i}) - v_i(x_i, x_{-i})\Big)\Big] f_{-i}(x_{-i}) dx_{-i} \end{aligned} \tag{134}$$

i.e.,

$$\begin{aligned} \int_{x_{-i}} \Big[q_i(x_i^{'}, x_{-i}) \Big(v_i(x_i, x_{-i}) - v_i(x_i^{'}, x_{-i})\Big)\Big] f_{-i}(x_{-i}) dx_{-i} &\leq U_i(x_i) - U_i(x_i^{'}) \\ \int_{x_{-i}} \Big[q_i(x_i, x_{-i}) \Big(v_i(x_i, x_{-i}) - v_i(x_i^{'}, x_{-i})\Big)\Big] f_{-i}(x_{-i}) dx_{-i} &\geq U_i(x_i) - U_i(x_i^{'}) \end{aligned} \tag{135}$$

Therefore, $q_i(x_i, x_{-i})$ is weakly increasing in x_i, and $U_i(x_i)$ has derivative

$$\frac{dU_i(x_i)}{dx_i} = \int_{x_{-i}} q_i(x_i, x_{-i})\Big[\frac{\partial v_i(x_i, x_{-i})}{\partial x_i}\Big] f_{-i}(x_{-i})dx_{-i} \tag{136}$$

integrate both sides yields

$$U_i(x_i) = \int_{x_{-i}} \int_0^{x_i} \Big[\frac{\partial v_i(s, x_{-i})}{\partial s}\Big] q_i(s, x_{-i})ds f_{-i}(x_{-i})dx_{-i} + U_i(0) \tag{137}$$

Since $q_i(x_i, x_{-i})$ is weakly increasing in x_i and $v_i(x_i, x_{-i})$ is strictly increasing in x_i, equation (137) implies that $U_i(0) \leq U_i(x_i)$ for all x_i. Therefore, $U_i(x_i) \geq 0$ for all $x_i \in [0, \bar{x}]$ only if $U_i(0) \geq 0$.

I next show that any mechanism (q, p) that satisfies the characterization in Lemma 7 must be Bayesian IC and IR. Since $q_i(x_i, x_{-i})$ is weakly increasing in x_i, $\frac{\partial v_i(s, x_{-i})}{\partial s} > 0$, and $U_i(x_i) = U_i(0) + \int_{x_{-i}} \int_0^{x_i} \Big[\frac{\partial v_i(s, x_{-i})}{\partial s}\Big] q_i(s, x_{-i})ds f_{-i}(x_{-i})dx_{-i}$, it is trivial that $U_i(x_i) \geq U_i(0)$ for all x_i, so $U_i(0) \geq 0$ implies Bayesian IR.

Suppose $x_i < x_i^{'}$, then

$$\begin{aligned}
U_i(x_i^{'}) &= U_i(x_i) + \int_{x_{-i}} \int_{x_i}^{x_i^{'}} q_i(s, x_{-i})\Big[\frac{\partial v_i(s, x_{-i})}{\partial s}\Big] ds f_{-i}(x_{-i})dx_{-i} \\
&\geq U_i(x_i) + \int_{x_{-i}} \int_{x_i}^{x_i^{'}} q_i(x_i, x_{-i})\Big[\frac{\partial v_i(s, x_{-i})}{\partial s}\Big] ds f_{-i}(x_{-i})dx_{-i} \\
&= U_i(x_i) + \int_{x_{-i}} \Big[q_i(x_i, x_{-i})\Big(v_i(x_i^{'}, x_{-i}) - v_i(x_i, x_{-i})\Big)\Big] f_{-i}(x_{-i})dx_{-i}
\end{aligned} \tag{138}$$

This directly implies Bayesian IC. □

The result of Lemma 7 leads to the following lemma that gives an expression of the seller's expected revenue in any Bayesian IC and IR mechanism.

Lemma 8. *For any Bayesian IC and IR mechanism that satisfy the conditions in*

Lemma 7, the ex-ante expected revenue is given by

$$ER = \int_x \sum_i \Big\{ q_i(x_i, x_{-i}) \Big\{ v_i(x_i, x_{-i}) - \frac{1 - F_i(x_i)}{f_i(x_i)} \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \Big\} \Big\} f(x) dx - \sum_i U_i(0) \tag{139}$$

Proof. The ex-ante expected payoff to an bidder i in any Bayesian IC and IR auction is

$$\begin{aligned}
E_{x_i}[U_i(x_i)] &= U_i(0) + \int_x \int_0^{x_i} q_i(s, x_{-i}) \Big[\frac{\partial v_i(s, x_{-i})}{\partial s} \Big] ds f(x) dx \\
&= U_i(0) + \int_{x_{-i}} \int_0^{\bar{x}} \int_0^{x_i} q_i(s, x_{-i}) \Big[\frac{\partial v_i(s, x_{-i})}{\partial s} \Big] ds f_i(x_i) dx_i f_{-i}(x_{-i}) dx_{-i} \\
&= U_i(0) + \int_{x_{-i}} \int_0^{\bar{x}} \int_s^{\bar{x}} q_i(s, x_{-i}) \Big[\frac{\partial v_i(s, x_{-i})}{\partial s} \Big] f_i(x_i) dx_i ds f_{-i}(x_{-i}) dx_{-i} \\
&= U_i(0) + \int_{x_{-i}} \int_0^{\bar{x}} \Big(1 - F_i(s)\Big) q_i(s, x_{-i}) \Big[\frac{\partial v_i(s, x_{-i})}{\partial s} \Big] ds f_{-i}(x_{-i}) dx_{-i} \\
&= U_i(0) + \int_{x_{-i}} \int_0^{\bar{x}} \Big(1 - F_i(x_i)\Big) q_i(x_i, x_{-i}) \Big[\frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \Big] dx_i f_{-i}(x_{-i}) dx_{-i} \\
&= U_i(0) + \int_x \Big[\frac{1 - F_i(x_i)}{f_i(x_i)} q_i(x_i, x_{-i}) \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \Big] f(x) dx
\end{aligned} \tag{140}$$

The ex-ante expected total surplus of the auction is given by

$$TS = \sum_i \int_x v_i(x_i, x_{-i}) q_i(x_i, x_{-i}) f(x) dx \tag{141}$$

The ex-ante expected revenue equals to the expected total surplus subtracted by the expected total payoff to all bidders:

$$\begin{aligned}
ER =& \sum_i \int_x v_i(x_i, x_{-i}) q_i(x_i, x_{-i}) f(x) dx \\
& - \sum_i \Big\{ U_i(0) + \int_x \Big\{ \frac{1 - F_i(x_i)}{f_i(x_i)} q(x_i, x_{-i}) \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \Big\} f(x) dx \Big\} \\
=& \sum_i \int_x \Big\{ q_i(x_i, x_{-i}) \times \Big\{ v_i(x_i, x_{-i}) - \frac{1 - F_i(x_i)}{f_i(x_i)} \frac{\partial v_i(x_i, x_{-i})}{\partial x_i} \Big\} \Big\} f(x) dx - \sum_i U_i(0)
\end{aligned} \tag{142}$$

□

According to the definition of $MR_i(x_i, x_{-i})$ with independent signals, the seller's problem is to maximize

$$ER = \int_x \sum_i \Big\{ q_i(x_i, x_{-i}) \times MR_i(x_i, x_{-i}) \Big\} f(x)dx - \sum_i U_i(0) \tag{143}$$

subject to no reserve price, $U_i(0) \geq 0$, $q_i(x_i, x_{-i})$ being weakly increasing in x_i, and the feasibility constraint. Since $U_i(0)$ is a constant, it is optimal to set $U_i(0) = 0$. The expected revenue is maximized by assigning higher q_i to bidders with higher $MR_i(x_i, x_{-i})$. Under this allocation rule, the constraint that $q_i(x_i, x_{-i})$ being weakly increasing in x_i is satisfied if $MR_i(x_i, x_{-i})$ is strictly increasing in x_i. Therefore, given that $MR_i(x_i, x_{-i})$ is strictly increasing in x_i, the optimal allocation rule $q(x_i, x_{-i})$ is given by

$$q_i^*(x_i, x_{-i}) = \begin{cases} \alpha_k & \text{if} \quad \hat{X}^k(x_{-i}) \leq x_i < \hat{X}^{k-1}(x_{-i}) \\ 0 & \text{if} \quad x_i < \hat{X}^K(x_{-i}) \end{cases} \tag{144}$$

in which $[\hat{X}^k(x_{-i}), \hat{X}^{k-1}(x_{-i})]$ is the interval of value that bidder i's signal x_i can take such that bidder i has the k-th highest $MR_i(x_i, x_{-i})$ given opponents' report x_{-i}.

The Bayesian IC and IR conditions given in Lemma 7 can be jointly written as

$$\int_{x_{-i}} \Big\{ q_i(x)v_i(x) - \int_0^{x_i} q_i(s, x_{-i}) \frac{\partial v_i(s, x_{-i})}{\partial s} ds - p_i(x) \Big\} f_{-i}(x_{-i})dx_{-i} = U_i(0) \geq 0 \tag{145}$$

for all bidder i. Choose $p_i^*(x) = q_i^*(x)v_i(x) - \int_0^{x_i} q_i^*(s, x_{-i}) \frac{\partial v_i(s,x_{-i})}{\partial s} ds$, then $p_i^*(x)$ satisfies the joint constraint. Therefore, (q^*, p^*) is the optimal position auction subject to no reserve price among all Bayesian IC and IR mechanisms when bidders have independent signals. □

Proof of Proposition 9:

Proof. Under regularity conditions **R1-R3**, it is trivial that given any profile of signals, the rank ordering of signals is equivalent to the rank ordering of values $v_i(x_i, x_{-i})$ as well as the rank ordering of marginal revenues $MR_i(x_i, x_{-i})$, so for any bidder i, given any opponents' report x_{-i}, we must have

$$\hat{x}^k(x_{-i}) = \hat{X}^k(x_{-i}), \quad \forall k \tag{146}$$

in which $\hat{x}^k(x_{-i})$ is the minimum value that bidder i's signal can take such that i has the k-th highest value $v_i(x_i, x_{-i})$ given x_{-i}, and $\hat{X}^k(x_{-i})$ is the minimum value that bidder i's signal can take such that i has the k-th highest marginal revenue $MR_i(x_i, x_{-i})$ given x_{-i}. Therefore, the allocation rule of the optimal auction (q^*, p^*) defined in Proposition 7 is the same as the allocation rule of the Generalized-VCG mechanism (q^V, p^V). Replacing $\hat{X}^k(x_{-i})$ by $\hat{x}^k(x_{-i})$ in the optimal auction (q^*, p^*) defined in Proposition 7 yields

$$q_i^*(x_i, x_{-i}) = \begin{cases} \alpha_k & \text{if} \quad \hat{x}^k(x_{-i}) \leq x_i < \hat{x}^{k-1}(x_{-i}) \\ 0 & \text{if} \quad x_i < \hat{x}^K(x_{-i}) \end{cases} \tag{147}$$

$$p_i^*(x_i, x_{-i}) = q_i^*(x_i, x_{-i}) v_i(x_i, x_{-i}) - \int_0^{x_i} q_i^*(s, x_{-i}) \frac{\partial v_i(s, x_{-i})}{\partial s} ds \tag{148}$$

I next substitute equation (147) into equation (148) to characterize the optimal payment rule p^*. Note that the term $q_i^*(s, x_{-i}) \times \frac{\partial v_i(s, x_{-i})}{\partial s}$ inside the integral in $p^*(x_i, x_{-i})$ is given by

$$q_i^*(s, x_{-i}) \times \frac{\partial v_i(s, x_{-i})}{\partial s} = \begin{cases} \alpha_k \frac{\partial v_i(s, x_{-i})}{\partial s} & \text{if} \quad \hat{x}^k(x_{-i}) \leq s < \hat{x}^{k-1}(x_{-i}) \\ 0 & \text{if} \quad s < \hat{x}^K(x_{-i}) \end{cases} \tag{149}$$

so the integral of $q_i^*(s,x_{-i})\frac{\partial v_i(s,x_{-i})}{\partial s}$ on $[0,x_i]$ is given by

$$\begin{aligned}
&\int_0^{x_i} q_i^*(s,x_{-i})\frac{\partial v_i(s,x_{-i})}{\partial s}ds \\
&=\begin{cases} \alpha_k \int_{\hat{x}^k(x_{-i})}^{x_i}\left[\frac{\partial v_i(s,x_{-i})}{\partial s}\right]ds + \sum_{j=k+1}^K\left\{\alpha_j \int_{\hat{x}^j(x_{-i})}^{\hat{x}^{j-1}(x_{-i})}\left[\frac{\partial v_i(s,x_{-i})}{\partial s}\right]ds\right\} & \text{if} \quad \hat{x}^k(x_{-i}) \le x_i < \hat{x}^{k-1}(x_{-i}) \\ 0 & \text{if} \quad x_i < \hat{x}^K(x_{-i}) \end{cases} \\
&=\begin{cases} \alpha_k\left[v_i(x_i,x_{-i}) - v_i(\hat{x}^k,x_{-i})\right] + \sum_{j=k+1}^K \alpha_j\left[v_i(\hat{x}^{j-1},x_{-i}) - v_i(\hat{x}^j,x_{-i})\right] & \text{if} \quad \hat{x}^k(x_{-i}) \le x_i < \hat{x}^{k-1}(x_{-i}) \\ 0 & \text{if} \quad x_i < \hat{x}^K(x_{-i}) \end{cases}
\end{aligned} \tag{150}$$

Substitute the optimal allocation rule q^* given in equation (147) and the integral given in equation (150) into the optimal payment rule p^* yields

$$p_i^*(x) = \begin{cases} \sum_{j=k}^K(\alpha_j - \alpha_{j+1})v_i(\hat{x}^j,x_{-i}) & \text{if} \quad x_i \in [\hat{x}^k(x_{-i}), \hat{x}^{k-1}(x_{-i})] \\ 0 & \text{if} \quad x_i < \hat{x}^K(x_{-i}) \end{cases} \tag{151}$$

which is equivalent to the Generalized-VCG payment rule. Therefore, under regularity conditions **R1-R3**, the Generalized-VCG mechanism is the optimal position auction subject to no reserve price among all ex-post IC and IR mechanisms.

I next compare the expected revenue of the Generalized-VCG mechanism to expected revenues of the GEA, the K-dimensional GSP auction and the K-dimensional VCG auction. Since expected revenue of the GEA is higher than the other two static auctions, showing that the Generalized-VCG mechanism yields higher expected revenue than the GEA is sufficient for proving the revenue ranking provided in Proposition 9.

The ex-ante expected price for the last position K in the Generalized-VCG mechanism and GEA are given by

$$\begin{aligned}
E\Big[p^{G-VCG,(K)}\Big] &= \alpha_K E\Big[v_i(Y_K,Y_1,\cdots,Y_{N-1})\Big|\{Y_{K-1} > X > Y_K\}\Big] \\
E\Big[p^{E,(K)}\Big] &= \alpha_K E\Big[v^{(K)}(Y_K,Y_K,\cdots,Y_{N-1})\Big|\{Y_{K-1} > X > Y_K\}\Big]
\end{aligned} \tag{152}$$

According to the Linkage Principle,

$$E\Big[p^{G-VCG,(K)}\Big] \geq E\Big[p^{E,(K)}\Big] \tag{153}$$

For any position $1 \leq k \leq K-1$, the expected price in the Generalized-VCG mechanism and GEA are given by

$$\begin{aligned}
E\Big[p^{G-VCG,(k)} - p^{G-VCG,(k+1)}\Big] &= (\alpha_k - \alpha_{k+1})E\Big[v_i(Y_k, Y_1, \cdots, Y_k, Y_{k+1}, \cdots, Y_{N-1})\Big|\{Y_{k-1} > X > Y_k\}\Big] \\
E\Big[p^{E,(k)} - p^{E,(k+1)}\Big] &= (\alpha_k - \alpha_{k+1})E\Big[v^{(k)}(Y_k, Y_k, Y_{k+1}, \cdots, Y_{N-1})\Big|\{Y_{k-1} > X > Y_k\}\Big]
\end{aligned} \tag{154}$$

Applying the Linkage Principle again yields

$$E\Big[p^{G-VCG,(k)}\Big] - E\Big[p^{G-VCG,(k+1)}\Big] \geq E\Big[p^{E,(k)}\Big] - E\Big[p^{E,(k+1)}\Big] \tag{155}$$

which implies that $E\Big[p^{G-VCG,(k)}\Big] \geq E\Big[p^{E,(k)}\Big]$ for all position k, so the expected revenue of the Generalized-VCG mechanism is higher than the expected revenue of the GEA, which is in turn higher than the expected revenue of K-dimensional VCG auction and K-dimensional GSP auction under affiliated signals.

In the special case of independent signals, it is trivial that

$$\begin{aligned}
E\Big[p^{G-VCG,(K)}\Big] &= E\Big[p^{E,(K)}\Big] \\
E\Big[p^{G-VCG,(k)} - p^{G-VCG,(k+1)}\Big] &= E\Big[p^{E,(k)} - p^{E,(k+1)}\Big]
\end{aligned} \tag{156}$$

which means $E\Big[p^{G-VCG,(k)}\Big] = E\Big[p^{E,(k)}\Big]$ for all position k, so the expected revenue of the Generalized-VCG mechanism is equivalent to the expected revenue of the GEA, which is in turn equivalent to the expected revenue of the K-dimensional VCG auction and the K-dimensional GSP auction under independent signals. It follows that the optimal revenue subject to no reserve price among all Bayesian IC and IR mechanisms is practically implementable by the GEA, the K-dimensional GSP auction, and the K-dimensional VCG auction under independent signals. □

References

[1] Abhishek, V. & Hosanagar, K. (2012). "Optimal Bidding in Multi-item Multi-slot Sponsored Search Auctions." Proceedings of the 13th ACM Conference on Electronic Commerce.

[2] Abrams, Z., Ghosh, A., & Vee, E. (2007). "Cost of Conciseness in Sponsored Search Auctions." In: Deng X., Graham F.C. (eds) Internet and Network Economics. WINE 2007. Lecture Notes in Computer Science, vol 4858. Springer, Berlin, Heidelberg.

[3] Aggarwal, G., Feldman, J., Muthukrishnan S., & Pal, M. (2008). "Sponsored Search Auctions with Markovian Users." Proceedings of the 4th Workshop on Ad Auctions.

[4] Athey, S. & Ellison, G. (2011). "Position Auctions with Consumer Search." *The Quarterly Journal of Economics*, Oxford University Press, vol. 126(3), 1213–1270.

[5] Ausubel, L. M. (1999). "A Generalized Vickrey Auction." Working Paper, University of Maryland.

[6] Ausubel, L. M. & Cramton P. C. (1999). "The Optimality of Being Efficient." World Bank Policy Research Working Paper No. 1985, University of Maryland.

[7] Ausubel, L. M. & Cramton P. C. (2004). "Vickrey Auctions with Reserve Pricing." *Economic Theory*, Vol. 23, Issue 3, 493–505.

[8] Benisch, M., Sadeh, N., & Sandholm, T. (2008). "The Cost of Inexpressiveness in Advertisement Auctions." Proceedings of ACM EC Workshop on Advertisement Auctions.

[9] Caragiannis, I., Kaklamanis, C., Kanellopoulos, P., Kyropoulou, M., Lucier, B., Leme, R. P., & Tardos, E. (2015). "Bounding the Inefficiency of Outcomes in Generalized Second Price Auctions." *Journal of Economic Theory,* 156 (2015), 343—388.

[10] Constantin, F., Malvika, R., Huang, C., & Parkes, D. C. (2011). "On Expressing Value Externalities in Position Auctions." Proceedings of the Twenty-fifth AAAI Conference on Artificial Intelligence (AAAI-11): August 7-11, 2011.

[11] Chen, Y. & He, C. (2011). "Paid Placement: Advertising and Search on the Internet." *The Economic Journal,* Vol. 121, Issue 556, 309–328.

[12] Dasgupta, P. & Maskin, E. (2000). "Efficient Auctions." *The Quarterly Journal of Economics*, Vol. 115, No. 2, 341–388.

[13] Dutting, P., Fischer, F. & Parkes, D. C. (2011). "Simplicity-Expressiveness Trade-offs in Mechanism Design." Proceedings of the 12th ACM conference on Electronic commerce, San Jose, California, USA.

[14] Deng, X. & Yu, J. (2009). "A New Ranking Scheme of the GSP Mechanism with Markovian Users." Proceedings of the 5th International Workshop on Internet and Network Economics, WINE 2009.

[15] Edelman, B., Ostrovsky, M. & Schwarz, M. (2007). "Internet Advertising and the Generalized Second-Price Auction: Selling Billions of Dollars Worth of Keywords." *The American Economic Review*, 97 (2007), 242–259.

[16] Edelman, B. & Schwarz, M. (2010). "Optimal Auction Design and Equilibrium Selection in Sponsored Search Auctions." *The American Economic Review*, Vol. 100, no. 2, 597–602.

[17] Farboodi, M. & Jafarian, A. (2013). "Optimal Revenue Maximizing Mechanisms in Common-Value Position Auctions." Working Paper.

[18] Fershtman, D. & Pavan, A. (2016). "Matching Auctions." `http://faculty.wcas.northwestern.edu/~apa522/DM-October-9.pdf`

[19] Ghosh, A. & Mahdian, M. (2008). "Externalities in Online Advertising." Proceedings of the 17th international conference on World Wide Web, WWW 2008, pp. 161–168.

[20] Goldman, M. & Rao, J. M. (2014). “Experiments as Instruments: Heterogeneous Position Effects in Sponsored Search Auctions.” Working Paper, University of California San Diego.

[21] Gomes, R. & Sweeney, K. (2014). “Bayes-Nash Equilibria of the Generalized Second-price Auction.” *Games and economic behavior*, Vol. 86, 421–437.

[22] Hummel, P. & McAfee, P. R. (2014). “Position Auctions with Externalities and Brand Effects.” *Web and Internet Economics,* Vol. 8877, 417–422.

[23] Izmalkov, S., Khakimova, D. & Romanyuk, G. (2016). “Position Auctions with Endogenous Supply.” Working Paper.

[24] Jehiel, P. & Moldovanu, B. (2001). “Efficient Design with Interdependent Valuations.” *Econometrica*, Vol. 69, Issue 5, 1237–1259.

[25] Kempe, D. & Mahdian, M. (2008). “A Cascade Model for Externalities in Sponsored Search.” Proceedings of the 4th Workshop on Internet Ad Auctions.

[26] Kominers, S. D. (2009). “Dynamic Position Auctions with Consumer Search.” Lecture Notes in Computer Science, vol 5564. Springer, Berlin, Heidelberg.

[27] Li, Y. (2017). “Approximation in Mechanism Design with Interdependent Values.” *Games and Economic Behavior*, Vol. 103, 225-253.

[28] Lu, Z. & Riis, C. (2016). “Bayes-Nash Equilibria in Generalized Second Price Auctions with Allocative Externalities.” Working Paper, Center for Research in Economics and Management, CREAM Publication No. 1, 2016.

[29] Lucier, B. & Leme, R. P. (2011). “GSP Auctions with Correlated Types.” Proceedings of the 12th ACM Conference on Electronic Commerce (EC 2011).

[30] Milgrom, P. (2010). “Simplified Mechanisms with Applications to Sponsored Search and Package Auctions.” *Games and Economic Behavior*, 70 (1), 62–70.

[31] Milgrom, P. & Weber, R. J. (1982). “A Theory of Auctions and Competitive Bidding.” *Econometrica*, 50 (5), 1089–1122.

[32] Myerson, R. B. (1981). "Optimal Auction Design." *Mathematics of Operations Research*, Vol. 6, No. 1.

[33] Perry, M. & Reny, P. J. (2002). "An Efficient Auction." *Econometrica*, Vol. 70, No.3, 1199–1212.

[34] Qin, T., Chen, W. & Liu, T. (2015). "Sponsored Search Auctions: Recent Advances and Future Directions." *ACM Transactions on Intelligent Systems and Technology (TIST)*, Vol.5, No.4.

[35] Roughgarden, T. & Tardos, E. (2015). "Do Externalities Degrade GSP's Efficiency?" Proceedings of the International Joint Conference on Artificial Intelligence, pp. 4207–4211.

[36] Roughgarden, T. & Talgam-Cohen, I. (2013). "Optimal and Near-Optimal Mechanism Design with Interdependent Values." Proceedings of the 14th ACM Conference on Electronic Commerce (EC), pp. 767—784.

[37] Ulku, L. (2013). "Optimal Combinatorial Mechanism Design." *Economic Theory*, 53, (2), pp. 473–498.

[38] Varian, H. R. (2007). "Position Auctions." *International Journal of Industrial Organization*, 25, 1163–1178.

[39] Vickrey, V. (1961). "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, 16(1), pp. 8-37.